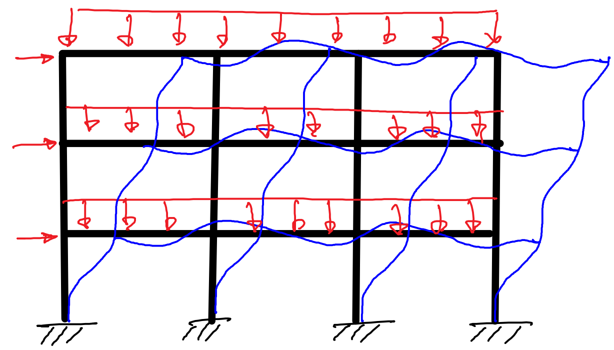
CE-474: Structural Analysis II

(Instructor: Arun Prakash)

What is Structural Analysis?

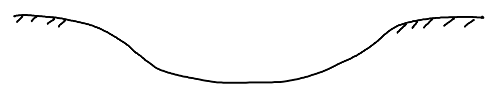


- Determine the "response" or behavior of a structure under some specified loads or combinations of loads
- Response includes: support reactions, internal stresses, and deformations / displacements
- It can also include: vibrations, stability of components / system, state of the constituent materials, occurrence of damage / failure etc.

Why is Structural Analysis needed and how does it fit into the "Big Picture?"

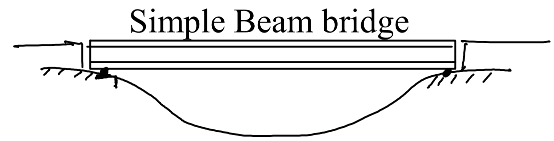
(Role of Structural Analysis in the Design Process)

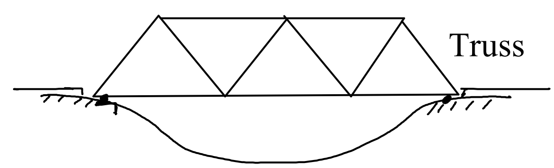
Consider a design project: say Bridge



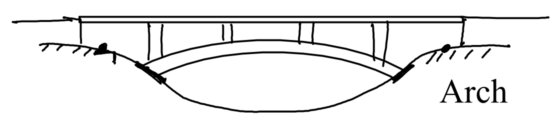
Things to consider:

- Type of bridge
- Loading classification
  - Traffic/Live Load,
  - EQ, Winds, Snow, Stream Ice
  - Temperature Thermal
  - Impact / Blast*
  - Fatigue
- Design Life ~50 years

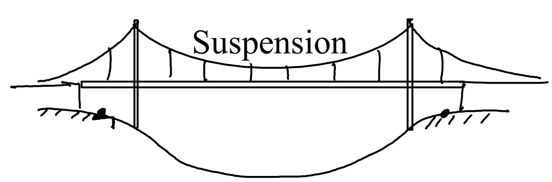




- Design Process
  - Assume a solution (based on experience, requirements)
  - Preliminary structural analysis
  - Refine the design
  - Check with detailed structural analysis

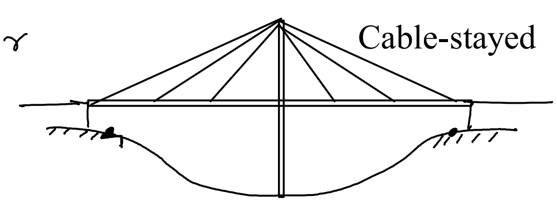


- Designs methodologies
  - ASD - Allowable Stress Design
  - LRFD - Load & Resistance factor design



$$\gamma L < \phi R$$

Load Safety factor   Strength reduction factor



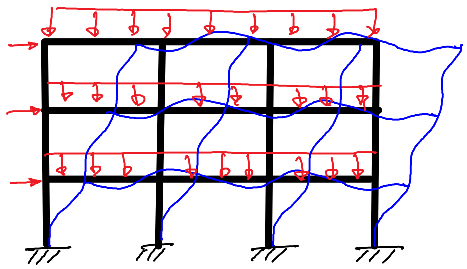
Note: Design is an <u>inverse</u> problem. It has many possible solutions.

- It can be framed as an optimization problem also:

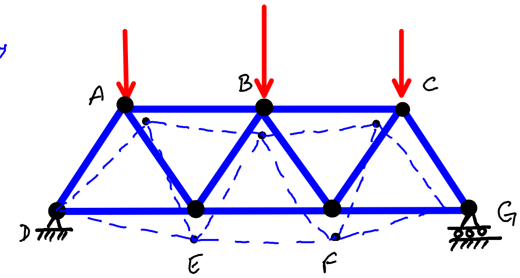
  Choose design parameters (criteria)
  - Material / Construction costs
  - Performance-based criteria

Problem Statement for Structural Analysis

Given:
- Structure Geometry,
- Loading,
- Material properties
- Support (boundary) conditions



Indeterminate



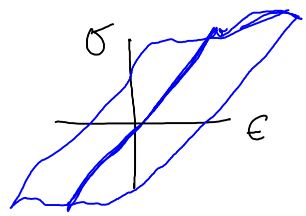
Determinate

To find: (unknowns at each and every point of the structure):
- External reactions
- Internal stresses and stress-resultants (axial force, shear force, bending moment)
- Deflections / displacements
- Strains
- Material response

Conditions / Governing Equations to satisfy using structural analysis:

1. Statics: Equilibrium of Forces and Moments
2. Compatibility of Deformations
3. Material Behavior
4. Boundary & Initial conditions

$\sum \vec{F} = \vec{0}$; $\sum \vec{M} = \vec{0}$



Methods of Structural Analysis
- Force (flexibility) method
- Displacement (stiffness) method

What about other types of structures?

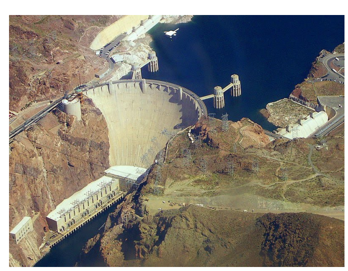


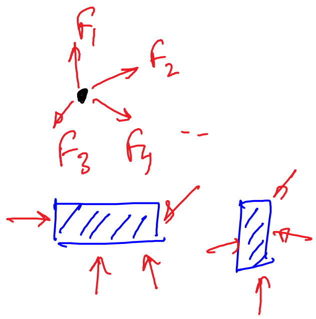
Plate and Shell structures

Large Continuum Structures

Complex assembly of components

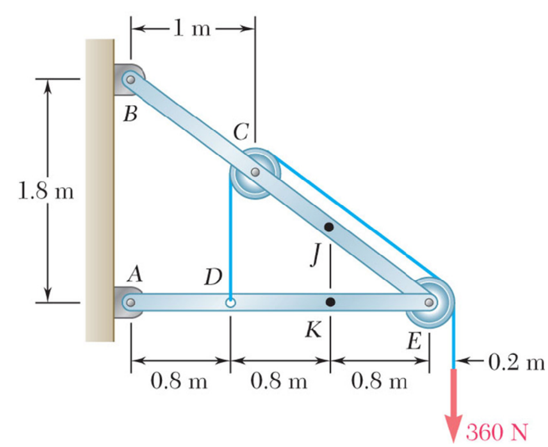
Statics: Equilibrium of Forces and Moments

Statics for:
- Point particles
- Rigid bodies
- Deformable bodies



$\sum \vec{F} = \vec{0}$

$\sum \vec{F} = \vec{0}$; $\sum \vec{M} = \vec{0}$



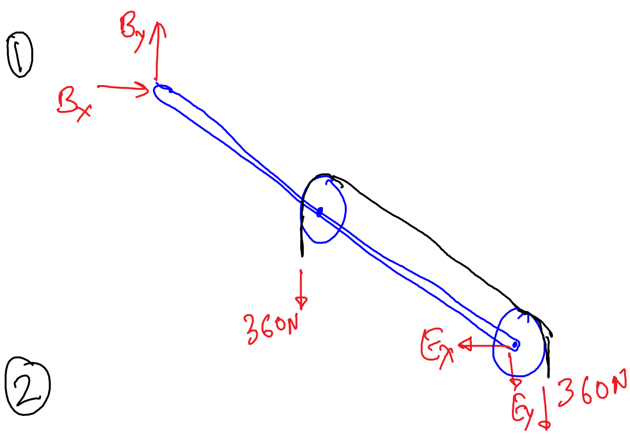
Example

Consider the frame shown.

Radius of both pulleys = 0.2 m

1. Is the frame statically determinate?
2. Draw the axial force, shear force and bending moment diagram for member AE

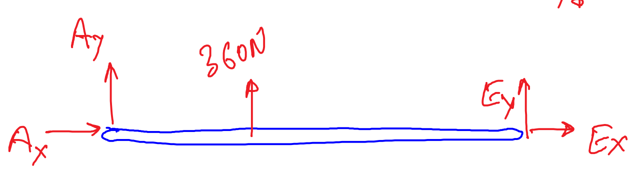
①

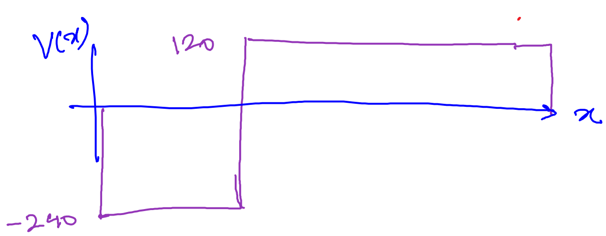


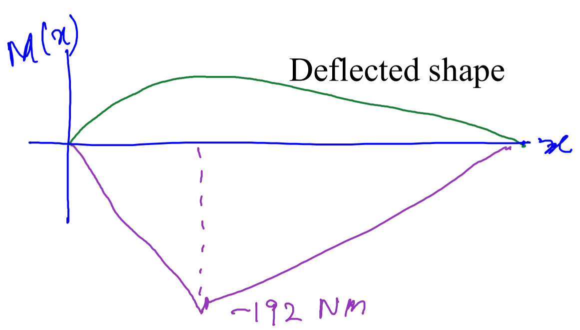
6 unknowns

6 equations : 3 eqns × 2 FBDs

②







FBD②: $\sum M_A = 0$

$360 \times 0.8 + E_y \times 2.4 = 0 \Rightarrow E_y = -\frac{360 \times 0.8}{2.4}$

$\Rightarrow E_y = \underline{-120\,N}$

$\sum F_y = 0 \Rightarrow A_y = -360 + 120 = \underline{-240\,N}$

Statics: Equilibrium of Deformable Bodies

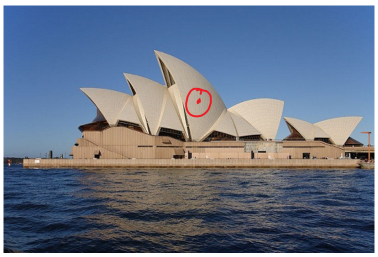
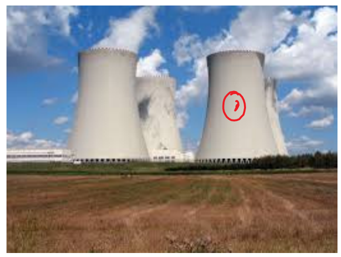
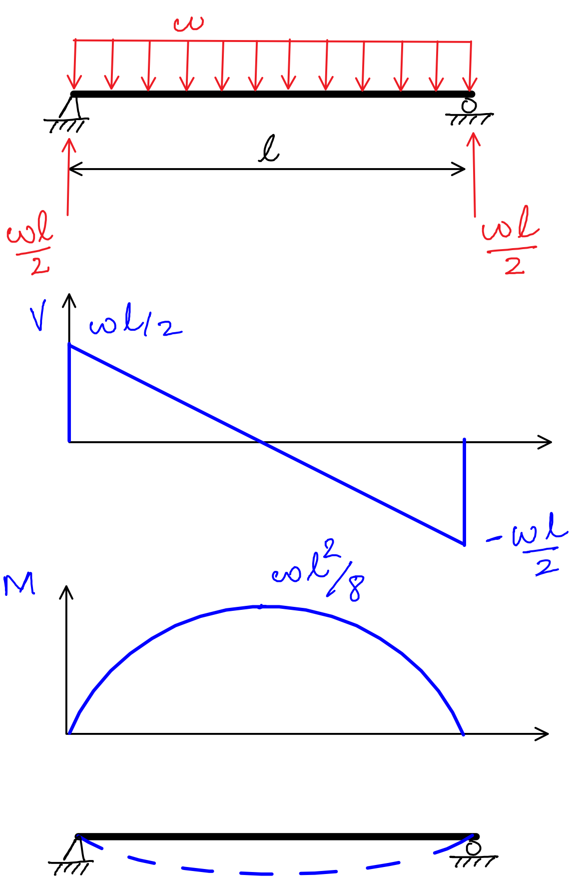



Plate and Shell structures

Large Continuum Structures

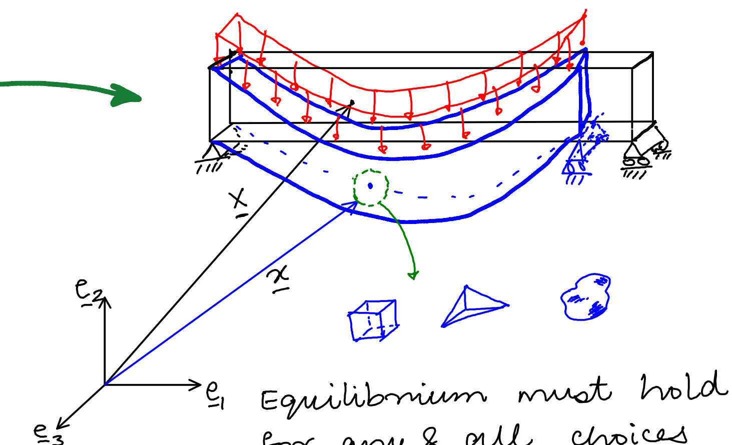
Complex assembly of components

Or even our beloved simply supported beam:

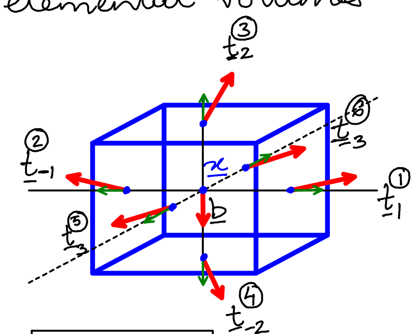


Deflection

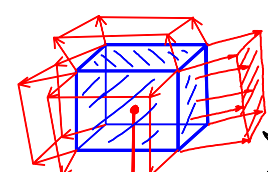
$$v(x) = -\frac{w x}{24 EI}\left(x^3 - 2Lx^2 + L^3\right)$$



Equilibrium must hold for any & all choices of elemental volumes



(on all 6 faces)

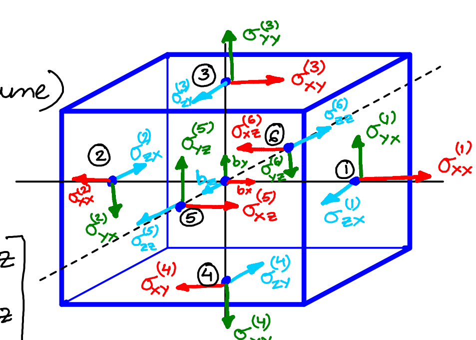


Traction

Distributed force per unit Area (need not be normal to the surface)

Body force (per unit volume)

$$\boxed{\underset{\sim}{t}_n = \underset{\sim}{\sigma}\, \underset{\sim}{n}}$$

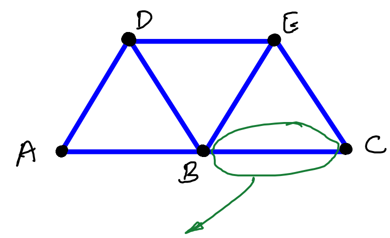


$$\underset{\sim}{\sigma} \rightarrow \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

Stress

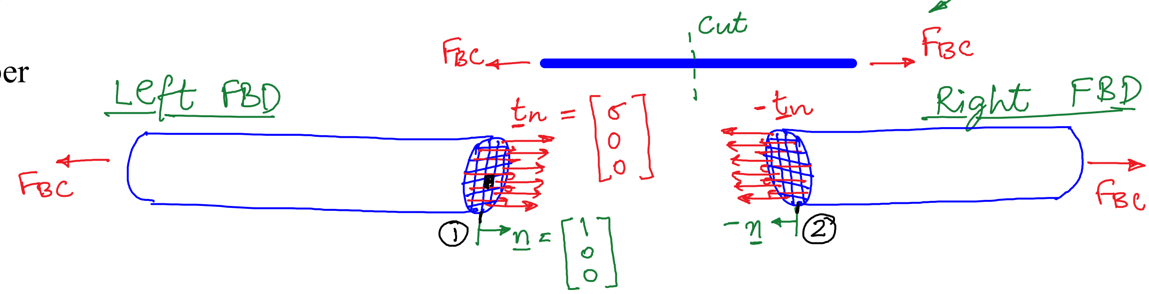
Concepts of Traction and Stress

In general,



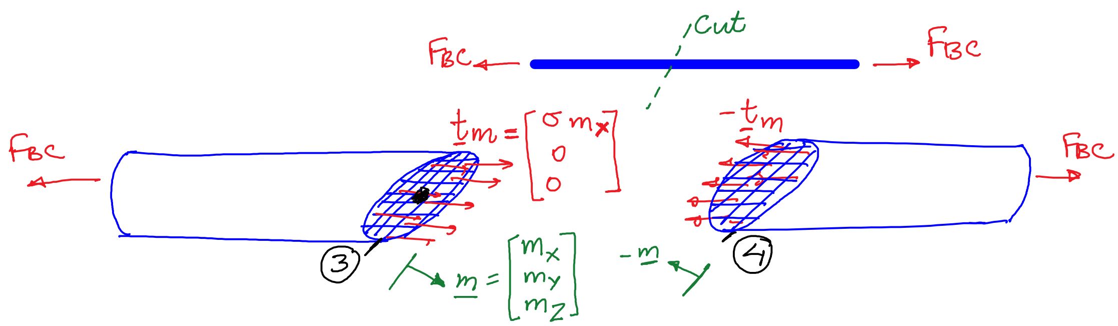
Traction is the distributed force per unit area acting at a point on any (external) surface of a body or a part of a body.
Traction is a **vector** represented with a 3x1 matrix in 3D.

Example: Truss member



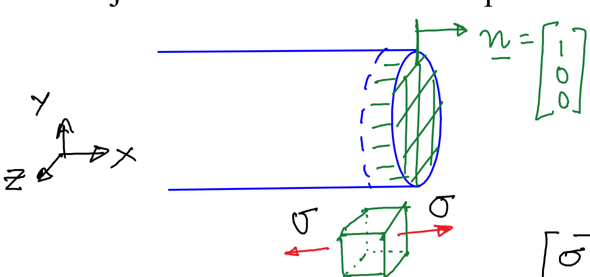
Note: $F_{BC} = \|\vec{F}_1\| = \|\vec{F}_2\|$ $\qquad \vec{F}_1 = \int_A \underset{\sim}{t_n}\, dA \qquad \vec{F}_2 = \int_A (-\underset{\sim}{t_n})\, dA$

Also, Alternate cut



Note: $F_{BC} = \|\vec{F}_3\| = \|\vec{F}_4\|$ $\qquad \vec{F}_3 = \int_{A_2} \underset{\sim}{t_m}\, dA \qquad \vec{F}_4 = \int_{A_2} (-\underset{\sim}{t_m})\, dA$

Stress is a physical quantity that completely characterizes the distributed internal forces per unit area that develop at a point within a body or a part of a body, at any orientation of the internal surface.
Stress is a **tensor** and is represented with a 3 x 3 matrix.
(Note: A tensor operates upon a vector to give another vector;
just like a 3×3 matrix multiplied with a 3×1 vector gives another 3×1 vector.)



$$\underset{\sim}{\sigma} \rightarrow \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{(x,y,z)}$$

Cauchy Stress-traction relationship

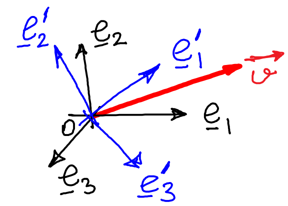
$$\boxed{\underset{\sim}{t_n} = \underset{\sim}{\sigma}\, \underset{\sim}{n}} \qquad \text{Note: } \boxed{\underset{\sim}{t_m} = \underset{\sim}{\sigma}\, \underset{\sim}{m}}$$

$$\begin{bmatrix} \sigma \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \sigma m_x \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} m_x \\ m_y \\ m_z \end{bmatrix}$$

Transformation of stress (Change of co-ordinate axes)

Recall that a vector is represented with a 3×1 matrix.
The components of this 3×1 vector depend upon the choice of axes.



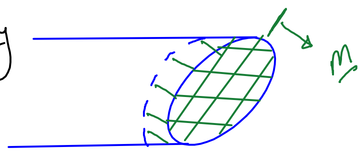
$$\vec{v} \rightarrow \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}_{(x,y,z)} \longrightarrow \begin{bmatrix} v_1' \\ v_2' \\ v_3' \end{bmatrix}_{(x',y',z')}$$

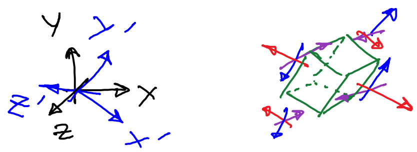
Note: $\{\underset{\sim}{v}'\} = [\underset{\sim}{Q}]\{\underset{\sim}{v}\}$

$$\begin{Bmatrix} v_1' \\ v_2' \\ v_3' \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}$$

where $Q_{ij} = (\underset{\sim}{e}_j \cdot \underset{\sim}{e}_i')$

Similarly





Thus, using the Cauchy-stress relationship

$$\underset{\sim}{t}_m = \underset{\sim}{\sigma}\, \underset{\sim}{m}$$

$$\underset{\sim}{\sigma} \rightarrow \begin{bmatrix} \sigma_{xx}' & \sigma_{xy}' & \sigma_{xz}' \\ \sigma_{yx}' & \sigma_{yy}' & \sigma_{yz}' \\ \sigma_{zx}' & \sigma_{zy}' & \sigma_{zz}' \end{bmatrix}_{(x',y',z')}$$

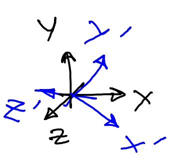
$$\begin{bmatrix} \sigma_{xx}' \\ \sigma_{yx}' \\ \sigma_{zx}' \end{bmatrix} = \begin{bmatrix} \sigma_{xx}' & \sigma_{xy}' & \sigma_{xz}' \\ \sigma_{yx}' & \sigma_{yy}' & \sigma_{yz}' \\ \sigma_{zx}' & \sigma_{zy}' & \sigma_{zz}' \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

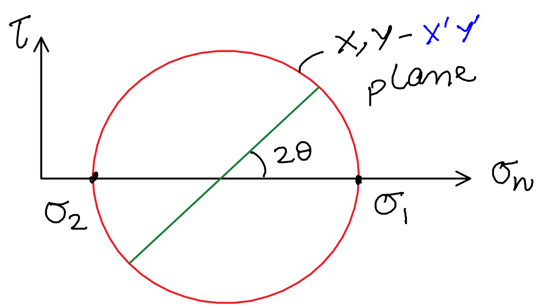
(x',y',z')

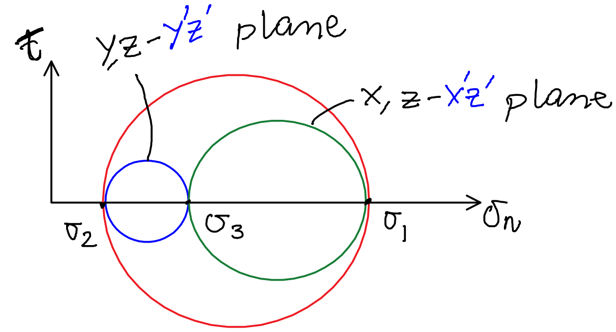
The transformed components of stress can be obtained as:

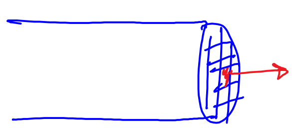
$$\begin{bmatrix} \sigma_{xx}' & \sigma_{xy}' & \sigma_{xz}' \\ \sigma_{yx}' & \sigma_{yy}' & \sigma_{yz}' \\ \sigma_{zx}' & \sigma_{zy}' & \sigma_{zz}' \end{bmatrix} = \begin{bmatrix} & & \\ & \underset{\sim}{Q} & \\ & & \end{bmatrix} \begin{bmatrix} \sigma_{xx} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} & & \\ & \underset{\sim}{Q} & \\ & & \end{bmatrix}^T$$

Also represented with Mohr's circles

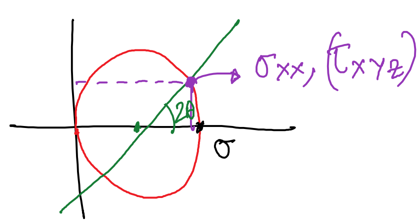








$$\begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

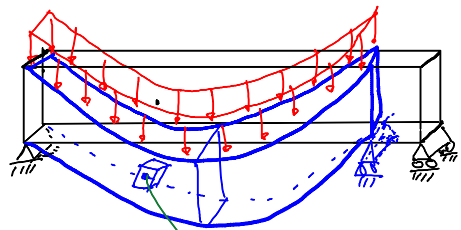


Equilibrium equations in terms of stresses

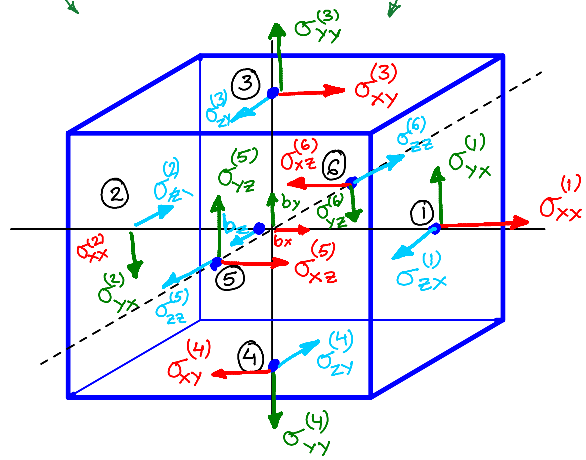
Using $\sum F_x = 0$

$\sum F_y = 0$

$\sum F_z = 0$



It is possible to show that:



$$\boxed{\begin{aligned}
&\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + b_x = 0\\
&\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + b_y = 0\\
&\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + b_z = 0
\end{aligned}}$$

at $\underline{\underline{ALL}}$ points.

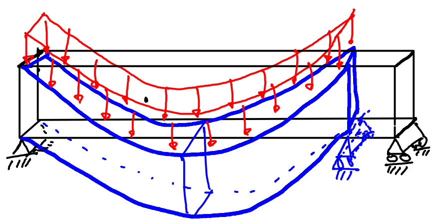
In <u>matrix form</u>, this may be expressed as:

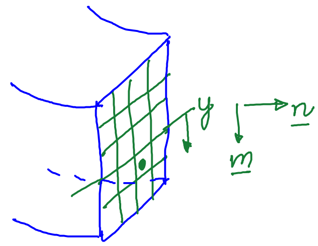
$$\left\{ \left\{ \partial/\partial x \quad \partial/\partial y \quad \partial/\partial z \right\} \begin{bmatrix} \sigma_{xx} & \sigma_{yx} & \sigma_{zx} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{zy} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix} \right\}^T + \begin{Bmatrix} b_x \\ b_y \\ b_z \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

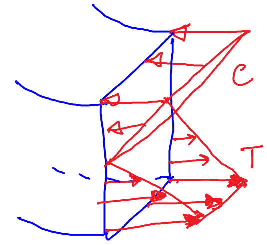
Equivalently: $\boxed{\operatorname{div} \underset{\sim}{\sigma} + \underline{b} = \underline{0}}$

Similarly $\sum \vec{M} = \vec{0} \Rightarrow \boxed{\underset{\sim}{\sigma} = \underset{\sim}{\sigma}^T}$

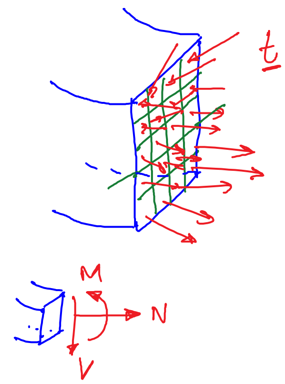
Stress Resultants and Equilibrium for Beams







Stress Resultants (Axial force, Shear force, Bending Moment)



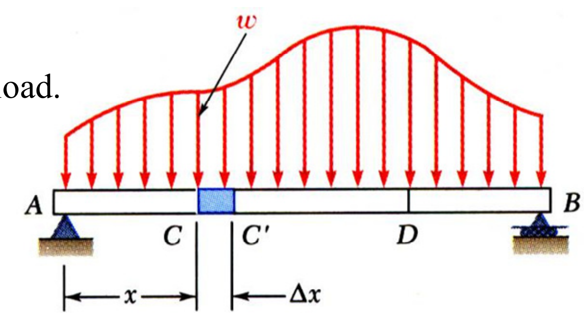
$$\text{Axial force } (N) = \int_A (\underline{t} \cdot \underline{n})\, dA$$

$$\text{Shear force } (V) = \int_A (\underline{t} \cdot \underline{m})\, dA$$

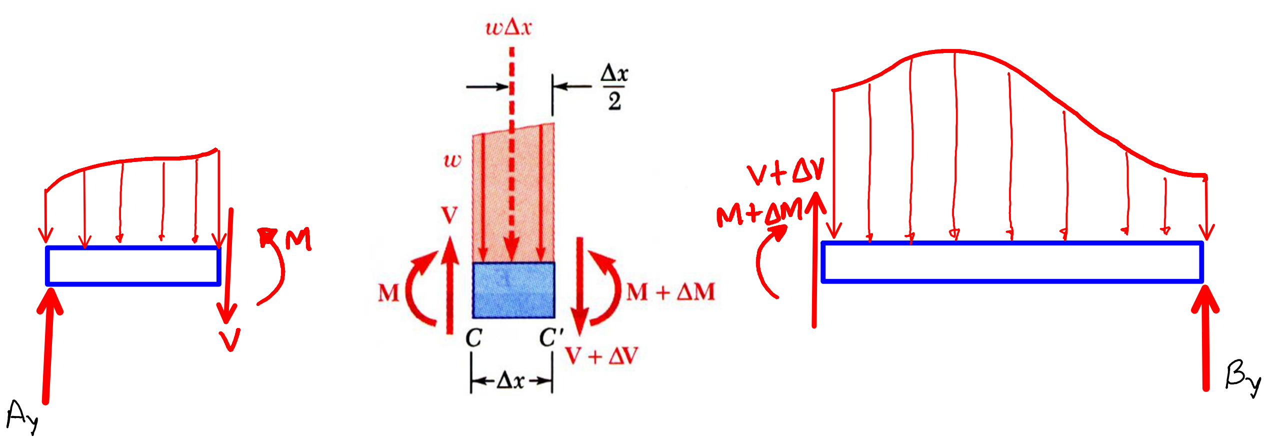
$$\text{Bending Moment } (M) = \int_A (\underline{t} \cdot \underline{n})\, y\, dA$$

Equilibrium in terms of Stress Resultants

Consider a small Δx length of **<u>any beam</u>** carrying a distributed load.



FBD of Δx element:

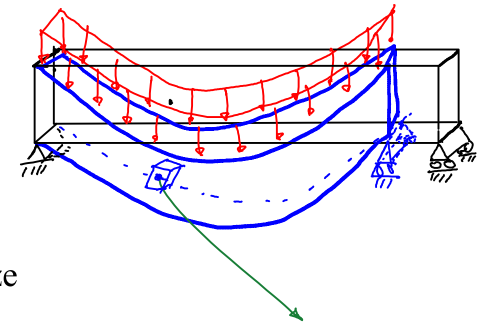


$$\Sigma F_y = 0 \Rightarrow \cancel{V} - w\Delta x - (\cancel{V} + \Delta V) = 0 \Rightarrow \boxed{w_{DN} = -\frac{dV}{dx}}$$

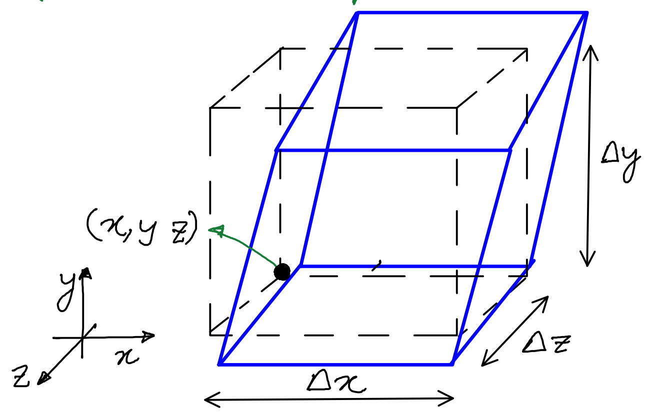
$$\Sigma M_{C'} = 0 \Rightarrow -M - V\Delta x + (M + \Delta M) + (w\Delta x)\frac{\Delta x}{2} = 0 \Rightarrow \boxed{V = \frac{dM}{dx}}$$

Concept of Strain

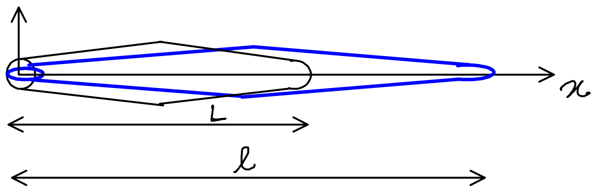
Under the action on external "loads", any deformable body undergoes changes in its shape and size. (*i.e.* it *deforms*).



Strain is a physical quantity that measures these changes in shape and size at a point in a body.

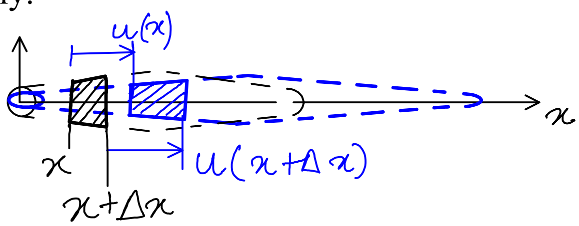


Example



$$\underset{(AVG)}{\bar{\epsilon}} = \frac{l - L}{L}$$

More precisely:



$$\epsilon(x) \equiv \lim_{\Delta x \to 0} \frac{u(x+\Delta x) - u(x)}{\Delta x}$$

$$\Rightarrow \boxed{\epsilon(x) = \frac{du(x)}{dx}}$$

In general Displacements:

$$\underset{\sim}{u}(\underset{\sim}{X}) = \begin{Bmatrix} u(x, y, z) \\ v(x, y, z) \\ w(x, y, z) \end{Bmatrix}$$

Strain (tensor):

$$\underset{\sim}{\epsilon}(\underset{\sim}{X}) \equiv \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix}$$

(under small displacements / deformations)

Displacement-strain relationships

Normal Strains

$$\boxed{\epsilon_{xx} = \frac{\partial u}{\partial x} \; ; \quad \epsilon_{yy} = \frac{\partial v}{\partial y} \; ; \quad \epsilon_{zz} = \frac{\partial w}{\partial z}}$$

Shear Strains

$$\epsilon_{xy} = \epsilon_{yx} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

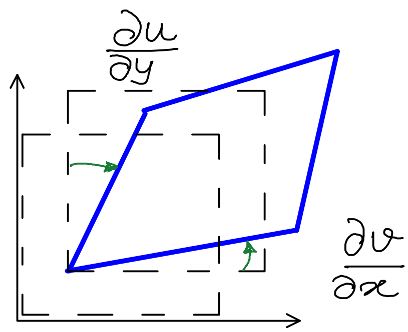
$$\epsilon_{yz} = \epsilon_{zy} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

$$\epsilon_{xz} = \epsilon_{zx} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

Shear angle

$$\gamma_{xy} = 2\,\epsilon_{xy}$$

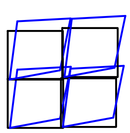
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$



Compatibility of strains

Meaning of compatibility of $\underset{\sim}{\epsilon}$:

Given $\underline{u}(\underset{\sim}{X}) \rightarrow \underset{\sim}{\epsilon}(\underset{\sim}{X})$ Automatically satisfied.



$\boxed{?} \leftarrow \underset{\sim}{\epsilon}(\underset{\sim}{X})$

$$\begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix}$$

Compatibility conditions:

$$\boxed{\frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2} = 2\,\frac{\partial^2 \epsilon_{xy}}{\partial x\,\partial y}}$$

Similarly 2 more equations

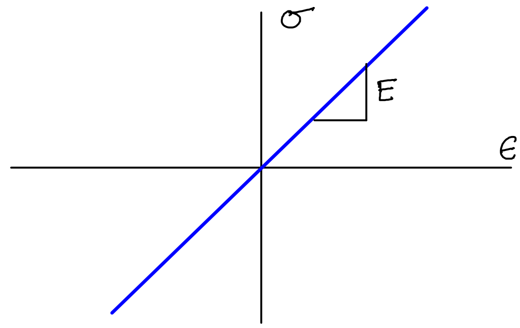
$$\boxed{\frac{\partial}{\partial x}\left(\frac{\partial \epsilon_{xy}}{\partial z} - \frac{\partial \epsilon_{yz}}{\partial x} + \frac{\partial \epsilon_{xz}}{\partial y}\right) = \frac{\partial^2 \epsilon_{xx}}{\partial y\,\partial z}}$$

Similarly 2 more equations

$\Rightarrow$ Total 6 equations of compatibility.

Stress-strain relationship (Material behavior)

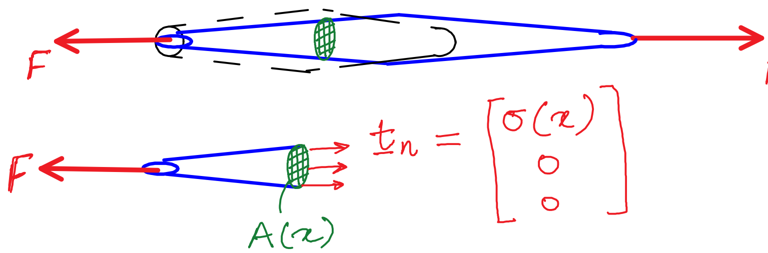
One of the simplest material behavior is characterized by the linear-elastic Hooke's ~~law~~ (model).



$$\sigma = E\epsilon$$

E : Young's modulus

In 1D



$$t_n = \begin{bmatrix} \sigma(x) \\ 0 \\ 0 \end{bmatrix}$$

$$\sigma(x) = \frac{F}{A(x)}$$

For 3D material behavior:

$$\underset{\sim}{\sigma} \rightarrow \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \quad \& \quad \underset{\sim}{\epsilon} \rightarrow \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix}$$

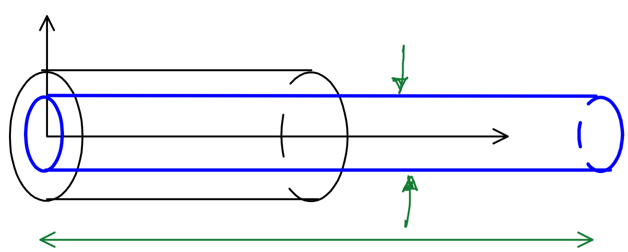
Using Voight Notation

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix}_{6\times1} = \frac{E}{(1+\nu)(1-2\nu)} \left[\begin{array}{ccc|ccc} (1-\nu) & \nu & \nu & & & \\ \nu & (1-\nu) & \nu & & 0 & \\ \nu & \nu & (1-\nu) & & & \\ \hline & & & \frac{1-2\nu}{2} & 0 & 0 \\ & 0 & & 0 & \frac{1-2\nu}{2} & 0 \\ & & & 0 & 0 & 1-2\nu \end{array}\right]_{6\times6} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ 2\epsilon_{xy} \\ 2\epsilon_{yz} \\ 2\epsilon_{zx} \end{Bmatrix}_{6\times1} \begin{matrix} \\ \\ \\ = \gamma_{xy} \\ = \gamma_{yz} \\ = \gamma_{zx} \end{matrix}$$

i.e. $\boxed{\underline{\sigma} = \underset{\sim}{D}\,\underline{\epsilon}}^{*}$ (* Note $\underset{\sim}{\sigma} \rightarrow \underline{\sigma}$; $\underset{\sim}{\epsilon} \rightarrow \underline{\epsilon}$)

In general a 3D linear-elastic material model is characterized by 2 material constants (properties):
E: Young's Modulus
ν : Poisson's Ratio

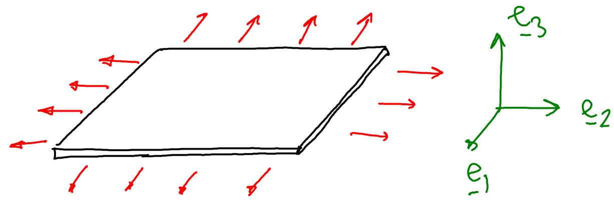
Note: $\nu = -\dfrac{\epsilon_{yy}}{\epsilon_{xx}}$



2D Plane Problems

- **Plane Stress**

$$\boxed{\sigma_{33} = 0 \quad ; \quad \sigma_{13} = \sigma_{31} = 0 \quad ; \quad \sigma_{23} = \sigma_{32} = 0}$$



Stress-strain relationship :

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ 2\epsilon_{xy} \end{Bmatrix}$$

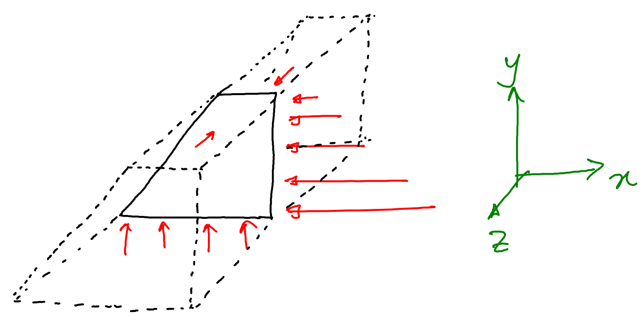
Note:

$$\epsilon_{zz} = \frac{-\nu}{E}\left(\sigma_{xx} + \sigma_{yy}\right)$$

ie $\boxed{\underline{\sigma} = \underset{\sim}{D}_{P\sigma}\, \underline{\epsilon}}$

- **Plane Strain**

$$\boxed{\epsilon_{33} = 0 \quad ; \quad \epsilon_{13} = \epsilon_{31} = 0 \quad ; \quad \epsilon_{23} = \epsilon_{23} = 0}$$



Stress-strain relationship :

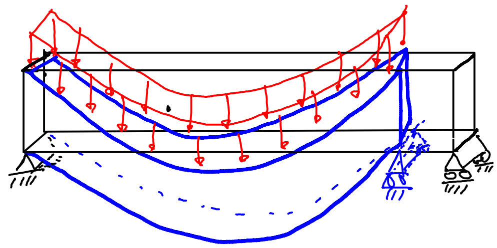
$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & 0 \\ \nu & (1-\nu) & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ 2\epsilon_{xy} \end{Bmatrix}$$

Note:

$$\sigma_{zz} = \nu\left(\epsilon_{xx} + \epsilon_{yy}\right)$$

ie $\boxed{\underline{\sigma} = \underset{\sim}{D}_{P\epsilon}\, \underline{\epsilon}}$

Final Problem Statement of Structural Analysis



Given

- geometry
- loads
- Material(s)

$$\underset{\sim}{\sigma} : \underset{\sim}{\sigma}(\underset{\sim}{\epsilon})$$

Find

- Displacement field $\underline{u}(\underline{x})$
- Stress field $\underset{\sim}{\sigma}(\underline{x})$
- Strain field $\underset{\sim}{\epsilon}(\underline{x})$

that satisfy the following conditions at $\boxed{\text{ALL}}$ points $\underline{x}$

- Equilibrium:

$$\text{div } \underset{\sim}{\sigma} + \underline{b} = \underline{0}$$

$$\underset{\sim}{\sigma} = \underset{\sim}{\sigma}^T$$

- Strain-displacement relationships:

$$\underset{\sim}{\epsilon} = \frac{1}{2}\left(\nabla \underline{u} + \nabla \underline{u}^T\right)$$

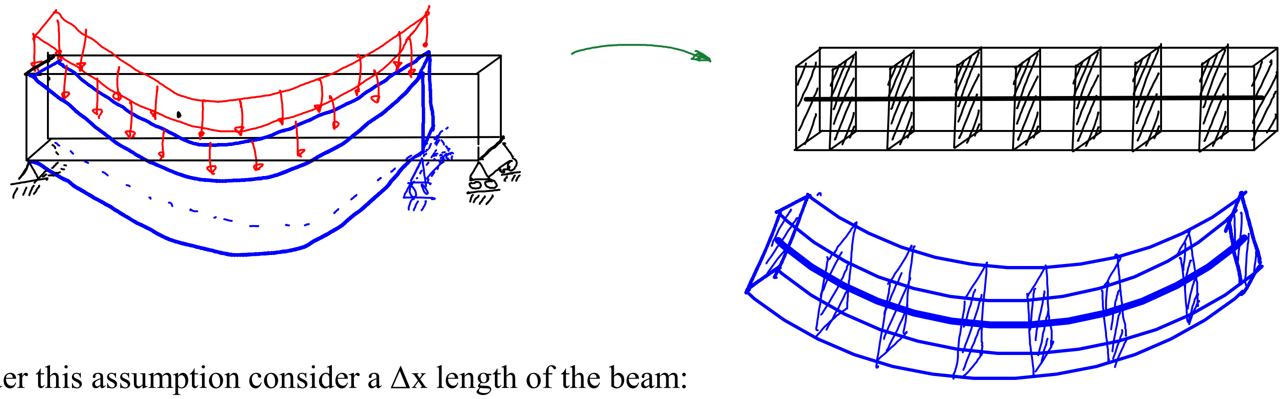
- Compatibility of strains
- Material relationships $\underset{\sim}{\sigma} : \underset{\sim}{\sigma}(\underset{\sim}{\epsilon})$
- Boundary conditions

Note:

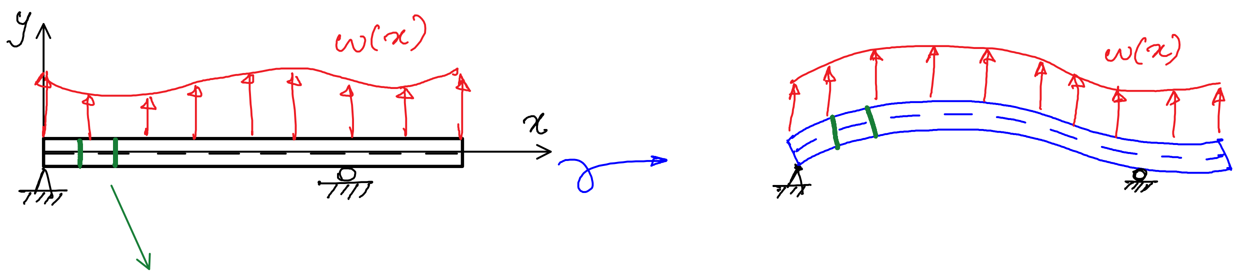
- For most practical problems, analytical (exact) solutions to the above system of PDEs, are not possible to obtain.
- Structural engineers resort to
  - make simplifying assumptions,
  - obtain approximate solutions to the above PDEs using numerical techniques like the finite element method.

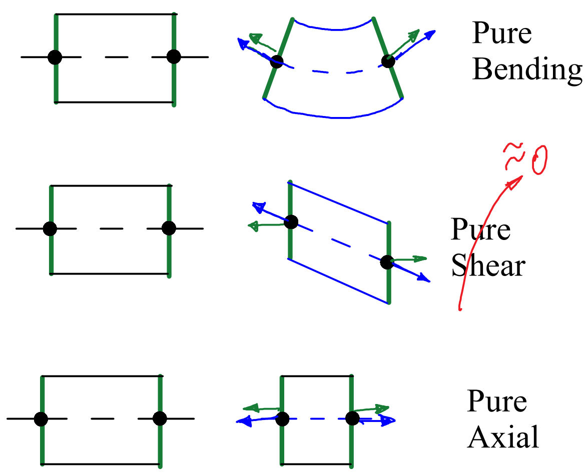
Structural Mechanics: Beam theory

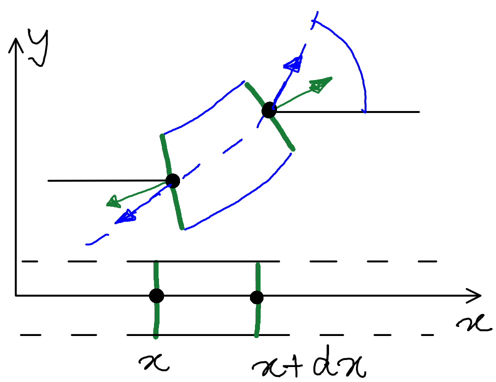
Kinematic Assumption: Assume that a beam consists of infinitely many RIGID CROSS- SECTIONS that are connected with a FLEXIBLE STRING (at their centroids).



Under this assumption consider a Δx length of the beam:







Actual response is some combination of all 3 "modes."

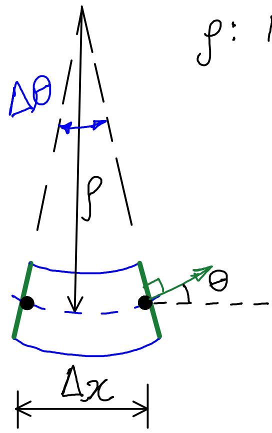
Note:

- The theory that accounts for all 3 modes of deformation above is called the Timoshenko Beam Theory. This theory is more general and is more suitable for "deep" / "short" beams i.e. beams whose depth/length ratio is less than 5.
  This is because for such beams bending deformations ane small and shear deformations contribute significantly to the response.

- An alternative theory which neglects the shear deformation is called the Bernoulli-Euler Beam theory. This theory is applicable only to "long" / "slender" beams whose depth / length ratio is greater than 10. For such beams, the bending deformations are much larger in comparison to the shear deformations, so neglecting shear deformations is justified.

Negligible Shear Deformations Assumption:

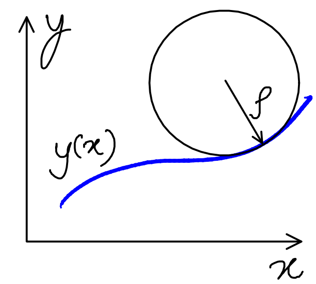
If we further assume that shear deformations are negligible, then:



ρ: Radius of Curvature of the centroidal curve

Using calculus, one can show:

$$\rho = \frac{\left[1 + \cancelto{\approx 0}{\left(\frac{dy}{dx}\right)^2}\right]^{3/2}}{\frac{d^2y}{dx^2}}$$



(Osculating Circle)

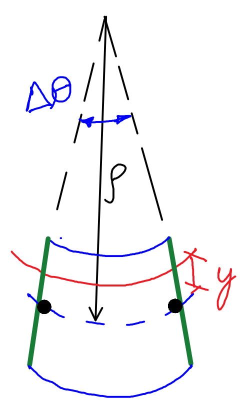
Note: $\rho\,\Delta\theta \approx \Delta x$

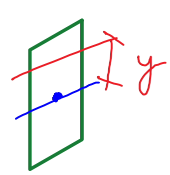
In the limit: $\lim_{\Delta x \to 0} \frac{\Delta\theta}{\Delta x} = \frac{1}{\rho} \Rightarrow$ $\boxed{\frac{d\theta}{dx} = \frac{1}{\rho} = \kappa}$ (Curvature)

Thus: $\kappa = \frac{1}{\rho} = \frac{\frac{d^2y}{dx^2}}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}} \Rightarrow$ $\boxed{\kappa = \frac{1}{\rho} \approx \frac{d^2y}{dx^2}}$

(Since θ is small $\Rightarrow \frac{dy}{dx} = \tan\theta \approx \theta$)

Furthermore:





Strain in fiber at distance "y":

$$\epsilon = \frac{(\rho - y)\,\Delta\theta - \rho\,\Delta\theta}{\rho\,\Delta\theta} \Rightarrow \boxed{\epsilon = -\frac{y}{\rho}}$$

Stress

$$\Rightarrow \sigma = E\epsilon \Rightarrow \boxed{\sigma = -\frac{Ey}{\rho}}$$

Moment

$$\Rightarrow M = -\int_A \sigma y\, dA = \frac{E}{\rho}\int_A y^2 dA \Rightarrow \boxed{M = \frac{EI}{\rho}}$$

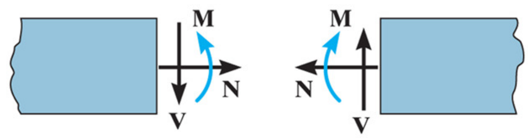
Finally this leads to the well known beam equations: $\boxed{\frac{M}{I} = \frac{-\sigma}{y} = \frac{E}{\rho}}$ and $\boxed{\kappa = \frac{M}{EI} = \frac{d^2y}{dx^2}}$

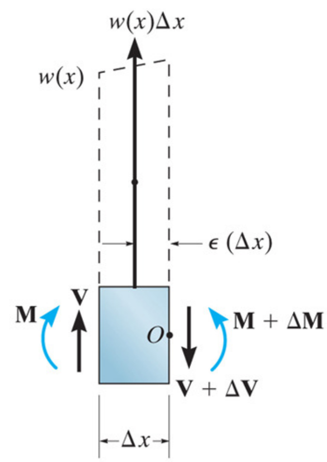
Calculating Deflections

(Ref: Chapter 8)

Sign Convention

Shear: Left Down & Right Up
Moment: Left Counter & Right Clockwise
(Smiley: positive)





$$\sum F_y = 0 \Rightarrow \cancel{V} + w\Delta x - (\cancel{V} + \Delta V) = 0 \Rightarrow \boxed{w_{UP} = \frac{dV}{dx}}$$

$$\sum M_{C'} = 0 \Rightarrow -M - V\Delta x + (M + \Delta M) - (w\Delta x)\frac{\Delta x}{2} = 0$$

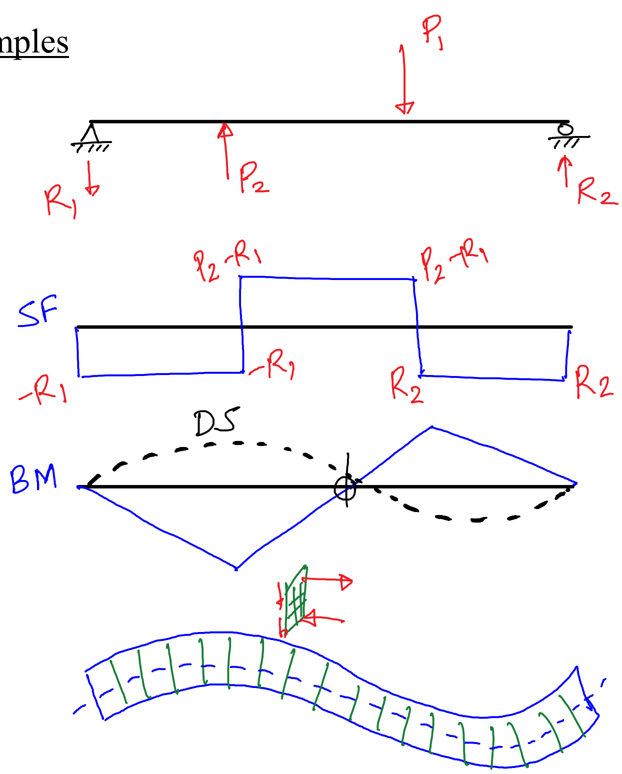
$$\Rightarrow \boxed{V = \frac{dM}{dx}}$$

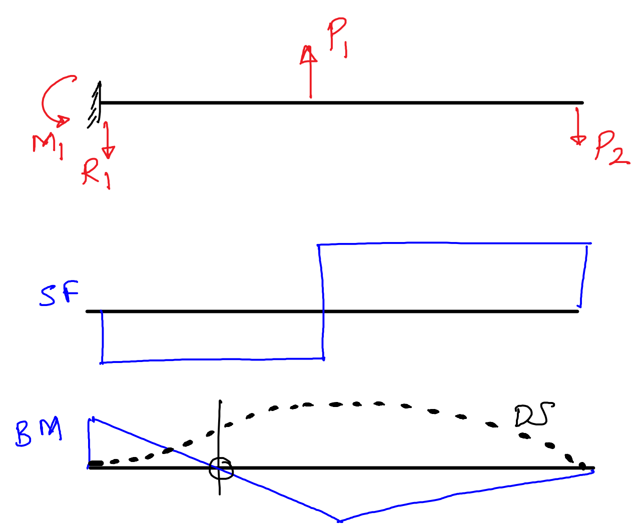
Deflections

Deflected shape (elastic curve) can be sketched by:

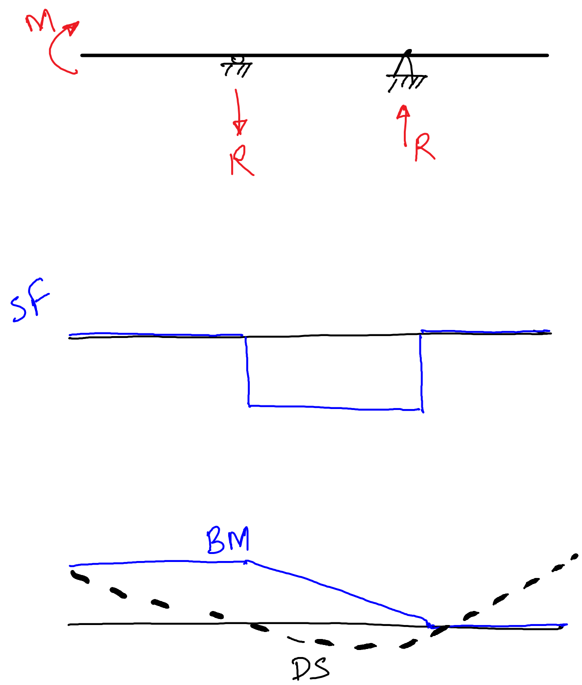
- Analyzing the loads and support conditions
- Bending moment diagram (using +/- curvatures and inflection points)

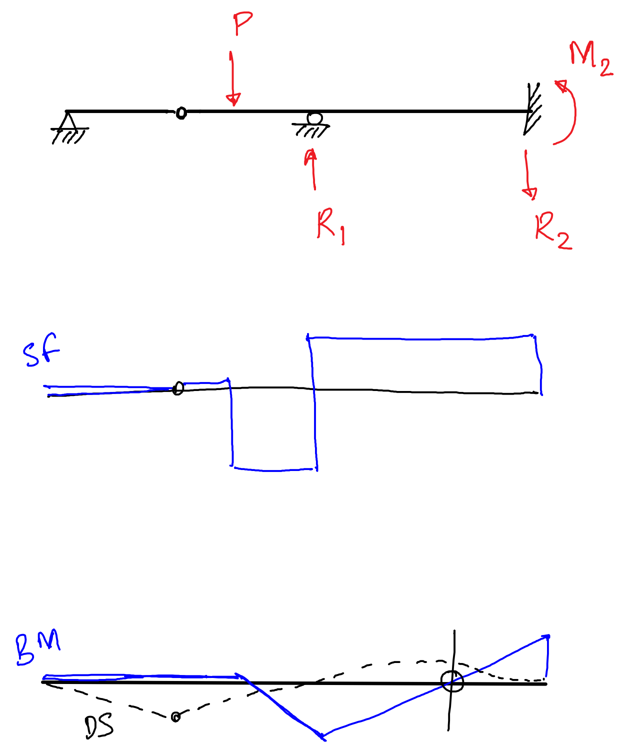
Examples





Examples

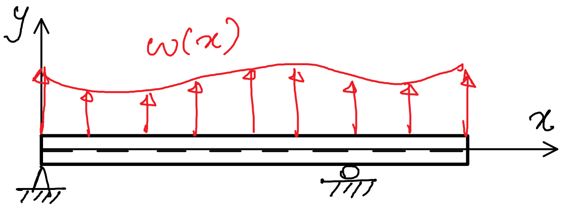


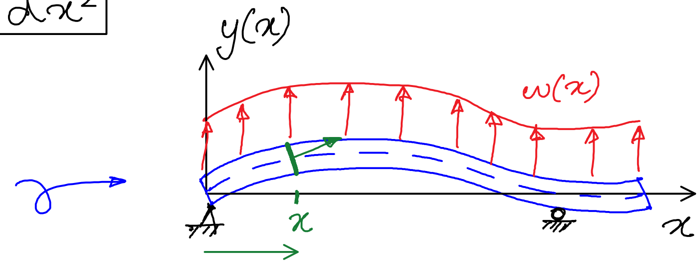


Double Integration

Deflected shapes of beams can also be calculated precisely by integrating the governing equation of beam equilibrium:

$$\boxed{\frac{M(x)}{EI} = \frac{d^2y}{dx^2}}$$





$$\Rightarrow \quad \frac{dy}{dx} = \int \frac{M}{EI}(x)\, dx + C_1 \qquad \text{(Slope at a point)}$$

$$\Rightarrow \quad y(x) = \iint \frac{M}{EI}(x)\, dx + C_1 x + C_2 \qquad \text{(Deflection)}$$

Note:

- Constants are evaluated using boundary and compatibility conditions at supports or interior points.
- If $M(x)$ diagram is discontinuous or has discontinuous changes in slope, then a single equation will not be possible for deflections and all segments would have to be integrated separately.

Example:

For $0 < x < L/4$

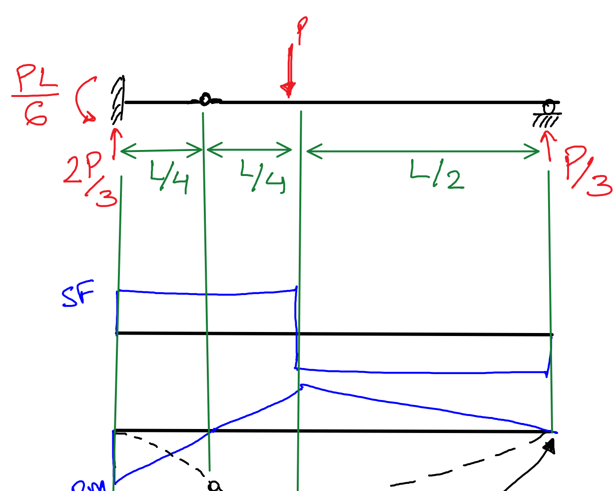
$$y(x) = \int_0^{L/4}\int_0^{L/4} \frac{M}{EI}(x)\, dx + C_1 x + C_2$$

For $L/4 < x < L/2$

$$y(x) = \int_{L/4}^{L/2}\int_{L/4}^{L/2} \frac{M}{EI}(x)\, dx + C_3 x + C_4$$

For $L/2 < x < L$

$$y(x) = \int_{L/2}^{L}\int_{L/2}^{L} \frac{M}{EI}(x)\, dx + C_5 x + C_6$$



⑥ $y(L) = 0$

① $\left.\frac{dy}{dx}\right|_{x=0} = 0$

② $y(0) = 0$

③ $y(L/4^-) = y(L/4^+)$

④ $\left.\frac{dy}{dx}\right|_{L/2^-} = \left.\frac{dy}{dx}\right|_{L/2^+}$

⑤ $y(L/2^-) = y(L/2^+)$

<u>Example:</u>

EXAMPLE 8.5

The beam in Fig. 8–13*a* is subjected to a load **P** at its end. Determine the displacement at *C*. *EI* is constant.

$\Sigma M_A = 0$

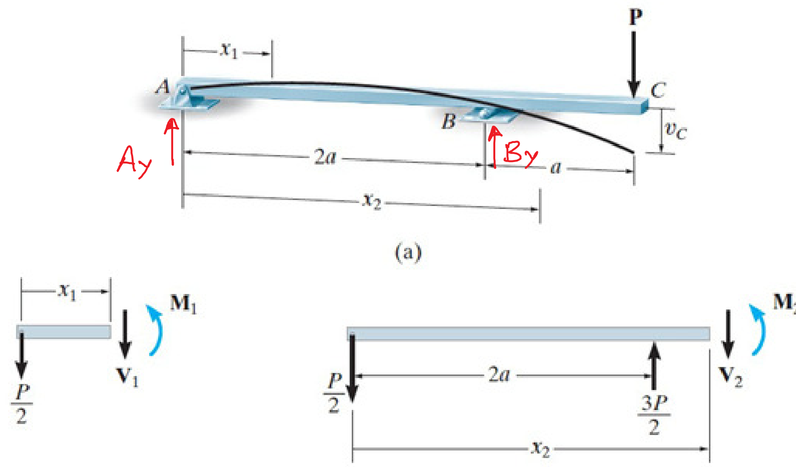
$\Rightarrow P \times 3a = B_y \times 2a$

$\Rightarrow \underline{B_y = \frac{3P}{2}}$

$\Sigma F_y = 0$

$\Rightarrow A_y = P - B_y$

$\Rightarrow \underline{A_y = -\frac{P}{2}}$



(a)

(b)

Fig. 8–13

EXAMPLE 8.5 CONTINUED

SOLUTION

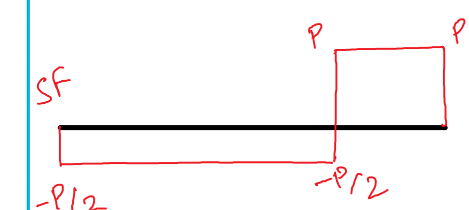
Elastic Curve. The beam deflects into the shape shown in Fig. 8–13*a*. Due to the loading, two *x* coordinates must be considered.

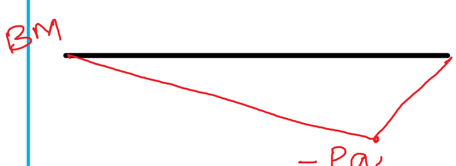
Moment Functions. Using the free-body diagrams shown in Fig. 8–13*b*, we have

$$M_1 = -\frac{P}{2}x_1 \qquad 0 \le x_1 \le 2a$$

$$M_2 = -\frac{P}{2}x_2 + \frac{3P}{2}(x_2 - 2a)$$

$$= Px_2 - 3Pa \qquad 2a \le x_2 \le 3a$$





Slope and Elastic Curve. Applying Eq. 8–4,

for x_1,
$$EI\frac{d^2v_1}{dx_1^2} = -\frac{P}{2}x_1$$

$$EI\frac{dv_1}{dx_1} = -\frac{P}{4}x_1^2 + C_1 \qquad (1)$$

$$EIv_1 = -\frac{P}{12}x_1^3 + C_1x_1 + C_2 \qquad (2)$$

EXAMPLE 8.5 CONTINUED

For x_2,
$$EI\frac{d^2v_2}{dx_2^2} = Px_2 - 3Pa$$

$$EI\frac{dv_2}{dx_2} = \frac{P}{2}x_2^2 - 3Pax_2 + C_3 \qquad (3)$$

$$EIv_2 = \frac{P}{6}x_2^3 - \frac{3}{2}Pax_2^2 + C_3x_2 + C_4 \qquad (4)$$

The *four* constants of integration are determined using *three* boundary conditions, namely, $v_1 = 0$ at $x_1 = 0$, $v_1 = 0$ at $x_1 = 2a$, and $v_2 = 0$ at $x_2 = 2a$, and *one* continuity equation. Here the continuity of slope at the roller requires $dv_1/dx_1 = dv_2/dx_2$ at $x_1 = x_2 = 2a$. (Note that continuity of displacement at B has been indirectly considered in the boundary conditions, since $v_1 = v_2 = 0$ at $x_1 = x_2 = 2a$.) Applying these four conditions yields

$v_1 = 0$ at $x_1 = 0$; $\qquad 0 = 0 + 0 + C_2$

$v_1 = 0$ at $x_1 = 2a$; $\qquad 0 = -\frac{P}{12}(2a)^3 + C_1(2a) + C_2$

$v_2 = 0$ at $x_2 = 2a$; $\qquad 0 = \frac{P}{6}(2a)^3 - \frac{3}{2}Pa(2a)^2 + C_3(2a) + C_4$

$\frac{dv_1(2a)}{dx_1} = \frac{dv_2(2a)}{dx_2}$; $\qquad -\frac{P}{4}(2a)^2 + C_1 = \frac{P}{2}(2a)^2 - 3Pa(2a) + C_3$

EXAMPLE 8.5 CONTINUED

Solving, we obtain

$$C_1 = \frac{Pa^2}{3} \qquad C_2 = 0 \qquad C_3 = \frac{10}{3}Pa^2 \qquad C_4 = -2Pa^3$$

Substituting C_3 and C_4 into Eq. (4) gives

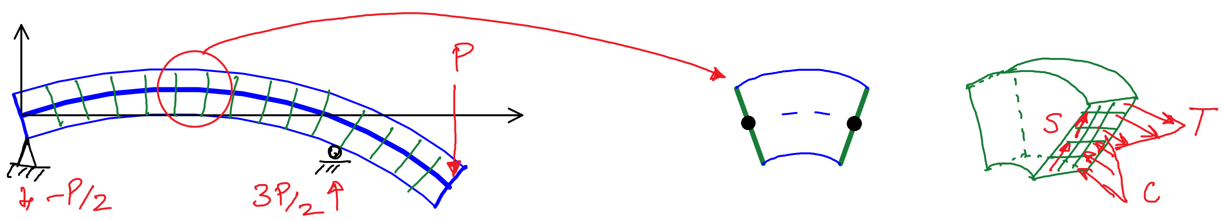
$$v_2 = \frac{P}{6EI}x_2^3 - \frac{3Pa}{2EI}x_2^2 + \frac{10Pa^2}{3EI}x_2 - \frac{2Pa^3}{EI}$$

$v_1(x) = \frac{-P}{12EI}x_1^3 + \frac{Pa^2}{3}x_1$

The displacement at C is determined by setting $x_2 = 3a$. We get

$$v_C = -\frac{Pa^3}{EI} \qquad \textit{Ans.}$$

In addition, overall deflected shape and internal stresses and stress resultants have also been calculated.



Moment-Area Theorems

An alternative to the double integration method is to use a semi-graphical method involving moment-area theorems.

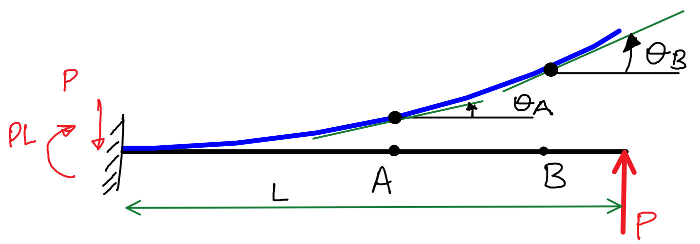
Note:

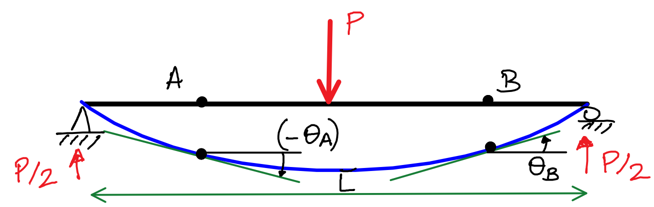
- Useful in situations where there are multiple segments of the beam (with different M/EI functions) that would lead to several boundary / continuity conditions to be solved for each segment.
- Usually this method doesn't give the slope or deflection directly.
  You have to use a geometrical construction in terms of the unknowns to solve for them.

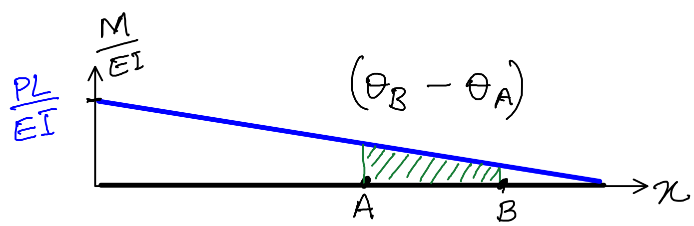
Theorem 1 (Single integration)

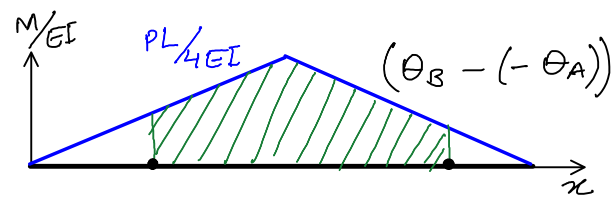
The change in slope of the deflected shape (elastic curve) of a beam between two points A and B is equal to the area under the M/EI diagram between these points.

$$\frac{d\theta}{dx} = \frac{M}{EI} \quad \Rightarrow \quad \theta(x_B) - \theta(x_A) = \int_A^B \frac{M}{EI}\,dx$$





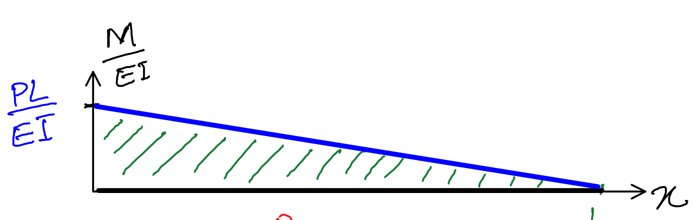




Note: θ is positive counter-clockwise.

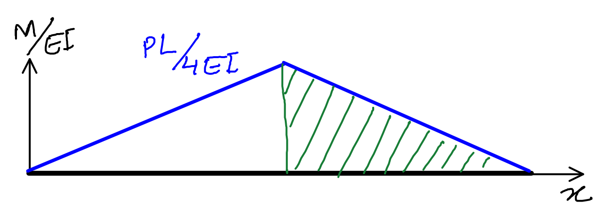
Examples

Slope at tip of cantilever with tip load:



$$\theta(L) - \cancelto{0}{\theta(0)} = \frac{1}{2}\,\frac{PL}{EI}\cdot L = \frac{PL^2}{2EI}$$

Slope at the ends of a simply supported beam with center point load:



$$\theta(L) - \cancelto{0}{\theta(L/2)} = \frac{1}{2}\,\frac{PL}{4EI}\cdot\frac{L}{2} = \frac{PL^2}{16EI}$$

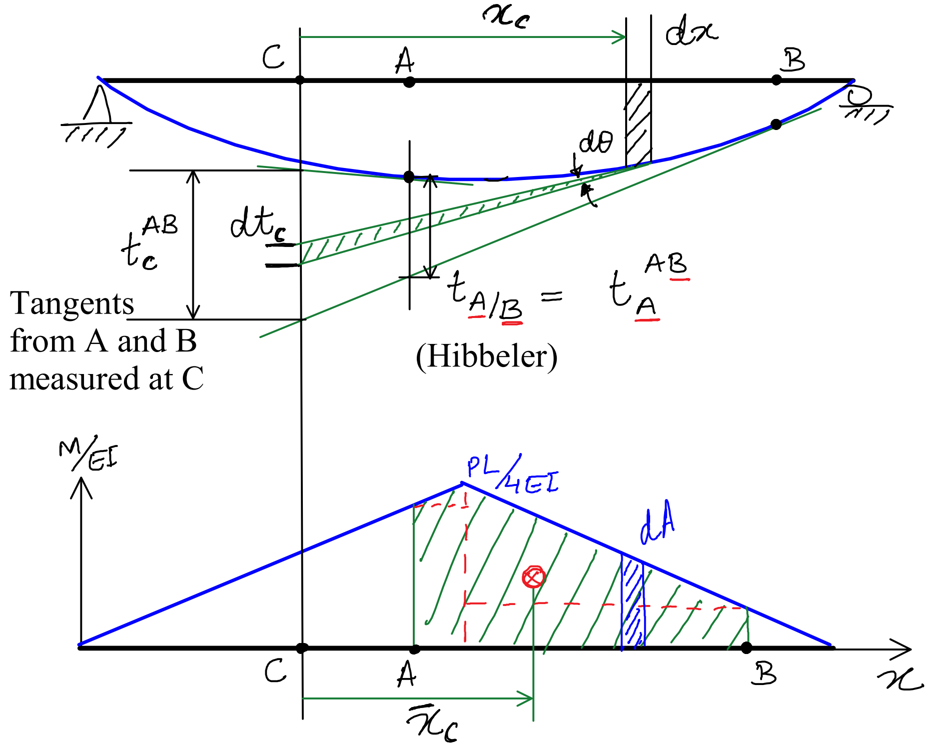
Theorem 2 ($dt_C = x_C . d\theta$)

For any two points A and B on a beam, the vertical distance between the tangents from A and from B at a third point C is equal to the 1st moment of area under the M/EI diagram between A and B taken about point C.

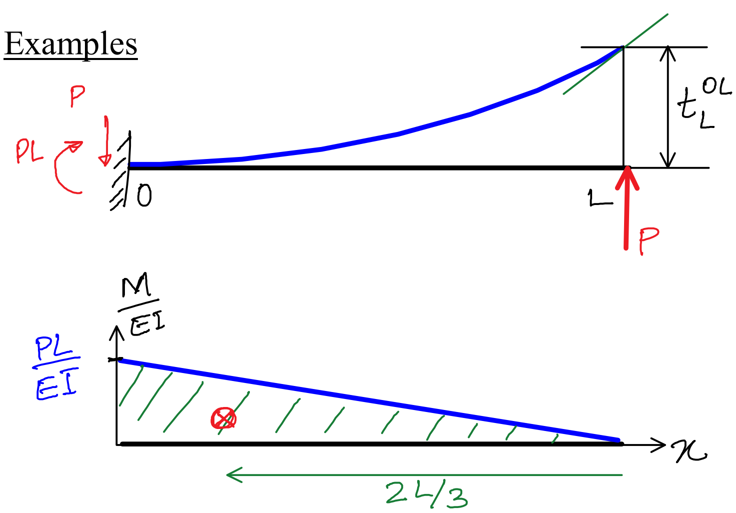
$$dt_C = x_C\, d\theta = x_C \left(\frac{d\theta}{dx}\right) dx = x_C \left(\frac{M}{EI}\right) dx$$

$$\Rightarrow \int_A^B dt_C = \int_A^B x_C \overbrace{\left(\frac{M}{EI}\right) dx}^{dA}$$

$$\boxed{t_C^{AB} = \bar{x}_C \int_A^B \underbrace{\left(\frac{M}{EI}\right) dx}_{dA}}$$

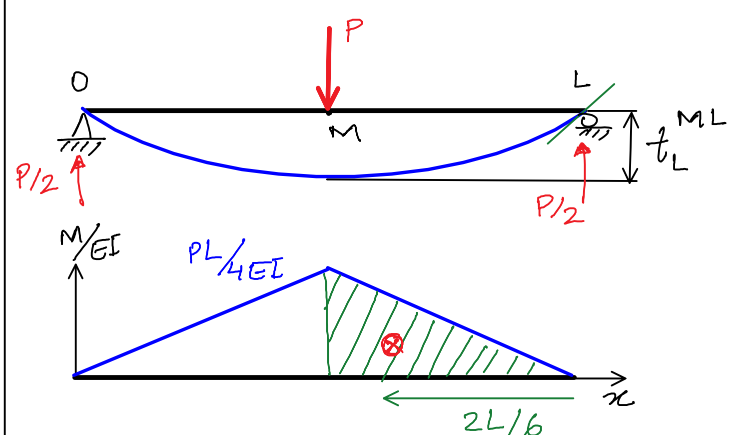


Examples



$$\Delta_L = t_L^{OL} = \frac{2L}{3} \cdot \left(\frac{1}{2} \frac{PL}{EI} \cdot L\right)$$

$$\Rightarrow \Delta_L = \frac{PL^3}{3EI}$$

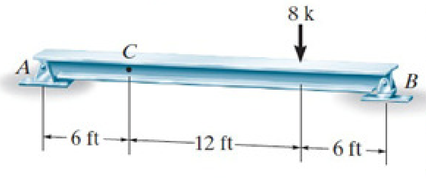


$$\Delta_M = t_L^{ML} = \frac{2L}{6} \cdot \left(\frac{1}{2} \frac{PL}{4EI} \cdot \frac{L}{2}\right)$$

$$\Rightarrow \Delta_M = \frac{PL^3}{48EI}$$

Example

EXAMPLE 8.9



(a)

Determine the slope at point C of the beam in Fig. 8–19a. $E = 29(10^3)$ ksi, $I = 600$ in^4.

$t_{B/A} = t_B^{AB}$

$\tan \phi = \dfrac{t_{B/A}}{L_{AB}}$

SOLUTION

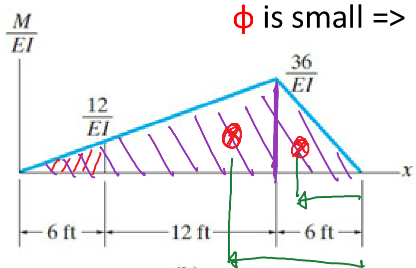
***M/EI* Diagram.** Fig. 8–19b.

Elastic Curve. The elastic curve is shown in Fig. 8–19c. We are required to find θ_C. To do this, establish tangents at A, B (the supports), and C and note that $\theta_{C/A}$ is the angle between the tangents at A and C. Also, the angle ϕ in Fig. 8–19c can be found using

ϕ is small =>

$\phi = t_{B/A}/L_{AB}$. This equation is valid since ~~$t_{B/A}$~~ is actually very small, so that $t_{B/A}$ can be approximated by the length of a circular arc defined by a radius of $L_{AB} = 24$ ft and sweep of ϕ. (Recall that $s = \theta r$.) From the geometry of Fig. 8–19c, we have

$$\theta_C = \phi - \theta_{C/A} = \frac{t_{B/A}}{24} - \theta_{C/A} \tag{1}$$



(b)

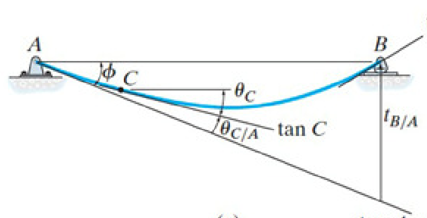
Moment-Area Theorems. Using Theorem 1, $\theta_{C/A}$ is equivalent to the area under the M/EI diagram between points A and C; that is,

$$\theta_{C/A} = \frac{1}{2}(6\text{ ft})\left(\frac{12\text{ k}\cdot\text{ft}}{EI}\right) = \frac{36\text{ k}\cdot\text{ft}^2}{EI}$$

EXAMPLE 8.9 CONTINUED

Applying Theorem 2, $t_{B/A}$ is equivalent to the moment of the area under the M/EI diagram between B and A about point B, since this is the point where the tangential deviation is to be determined. We have

$$t_{B/A} = \left[6\text{ ft} + \frac{1}{3}(18\text{ ft})\right]\left[\frac{1}{2}(18\text{ ft})\left(\frac{36\text{ k}\cdot\text{ft}}{EI}\right)\right]$$
$$+ \frac{2}{3}(6\text{ ft})\left[\frac{1}{2}(6\text{ ft})\left(\frac{36\text{ k}\cdot\text{ft}}{EI}\right)\right]$$
$$= \frac{4320\text{ k}\cdot\text{ft}^3}{EI}$$



(c)

Fig. 8–19

Substituting these results into Eq. 1, we have

$$\theta_C = \frac{4320\text{ k}\cdot\text{ft}^3}{(24\text{ ft})\,EI} - \frac{36\text{ k}\cdot\text{ft}^2}{EI} = \frac{144\text{ k}\cdot\text{ft}^2}{EI}$$

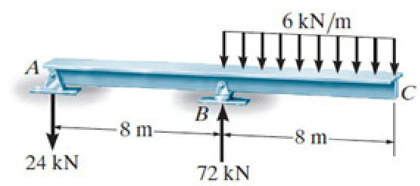
so that

$$\theta_C = \frac{144\text{ k}\cdot\text{ft}^2}{29(10^3)\text{ k/in}^2(144\text{ in}^2/\text{ft}^2)\,600\text{ in}^4(1\text{ ft}^4/(12)^4\text{ in}^4)}$$
$$= 0.00119\text{ rad} \qquad \textit{Ans.}$$

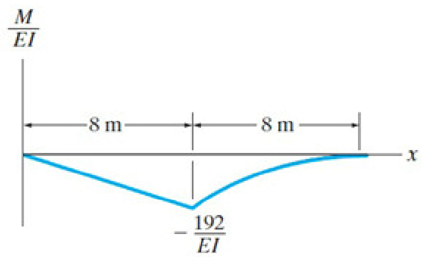
Note: This method does not give us an expression/equation for the slope or deflection at ALL points of the beam (as required by the general Problem statement of Structural Analysis), whereas the method of double integration does.
Nevertheless, one can find extremal values of slopes and deflections using this method, and usually these are sufficient for Structural Analysis and Design.

Example

EXAMPLE 8.11



(a)



(b)

Determine the deflection at point C of the beam shown in Fig. 8–21*a*. $E = 200$ GPa, $I = 250(10^6)$ mm^4.

SOLUTION

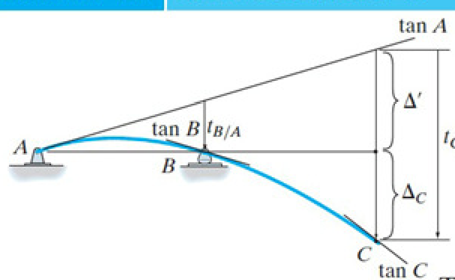
***M/EI* Diagram.** As shown in Fig. 8–21*b*, this diagram consists of a triangular and a parabolic segment.

Elastic Curve. The loading causes the beam to deform as shown in Fig. 8–21*c*. We are required to find Δ_C. By constructing tangents at A, B (the supports), and C, it is seen that $\Delta_C = t_{C/A} - \Delta'$. However, Δ' can be related to $t_{B/A}$ by proportional triangles, that is, $\Delta'/16 = t_{B/A}/8$ or $\Delta' = 2t_{B/A}$. Hence

$$\Delta_C = t_{C/A} - 2t_{B/A} \tag{1}$$

Moment-Area Theorem. We will apply Theorem 2 to determine $t_{C/A}$ and $t_{B/A}$. Using the table on the inside back cover for the parabolic segment and considering the moment of the *M/EI* diagram between A and C about point C, we have

EXAMPLE 8.11 CONTINUED



(c)

Fig. 8–21

$$t_{C/A} = \left[\frac{3}{4}(8\text{ m})\right]\left[\frac{1}{3}(8\text{ m})\left(-\frac{192\text{ kN}\cdot\text{m}}{EI}\right)\right] + \left[\frac{1}{3}(8\text{ m}) + 8\text{ m}\right]\left[\frac{1}{2}(8\text{ m})\left(-\frac{192\text{ kN}\cdot\text{m}}{EI}\right)\right]$$
$$= -\frac{11\,264\text{ kN}\cdot\text{m}^3}{EI}$$

The moment of the *M/EI* diagram between A and B about point B gives

$$t_{B/A} = \left[\frac{1}{3}(8\text{ m})\right]\left[\frac{1}{2}(8\text{ m})\left(-\frac{192\text{ kN}\cdot\text{m}}{EI}\right)\right] = -\frac{2048\text{ kN}\cdot\text{m}^3}{EI}$$

Why are these terms negative? Substituting the results into Eq. (1) yields

$$\Delta_C = -\frac{11\,264\text{ kN}\cdot\text{m}^3}{EI} - 2\left(-\frac{2048\text{ kN}\cdot\text{m}^3}{EI}\right)$$
$$= -\frac{7168\text{ kN}\cdot\text{m}^3}{EI}$$

Thus,

$$\Delta_C = \frac{-7168\text{ kN}\cdot\text{m}^3}{[200(10^6)\text{ kN/m}^2][250(10^6)(10^{-12})\text{ m}^4]}$$
$$= -0.143\text{ m} \qquad \textit{Ans.}$$

Conjugate Beam Analogy

The conjugate beam ~~method~~ analogy relies simply on the similarities between the governing equations of beam theory and those of beam equilibrium.

$$\boxed{w_{UP}(x) = \frac{dV(x)}{dx} \qquad V(x) = \frac{dM(x)}{dx}}$$

$$w_{UP}(x) \longleftrightarrow \frac{M}{EI}(x)$$

$$V(x) \longleftrightarrow \theta(x)$$

$$M(x) \longleftrightarrow y(x)$$

$$\boxed{\left(\frac{M(x)}{EI(x)}\right) = \frac{d\theta(x)}{dx} \qquad \theta(x) = \frac{dy(x)}{dx}}$$

Normally, for computation of slopes and displacements:

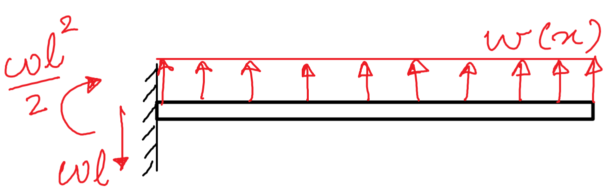
Loading $\xrightarrow{\text{(Statics)}}$ Reactions, Shear Force Diagram, Bending Moment Diagram $\xrightarrow{\text{(Double Integration)}}$ Slopes, Displacements

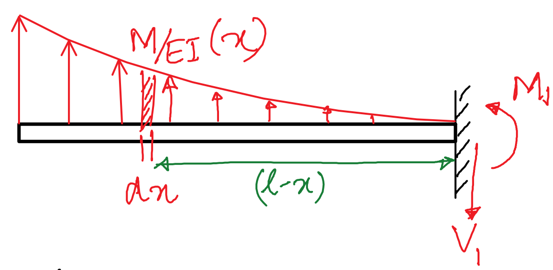
In the Conjugate Beam Analogy:

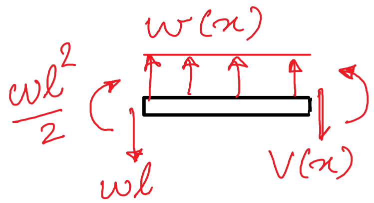
In addition, Boundary and Interior conditions

REAL	CONJUGATE
	$w_c(x) = \frac{M}{EI}(x)$ ←
$\theta(x)$ ←	$V_c(x)$ ✓
$y(x)$ ←	$M_c(x)$ ✓

Example

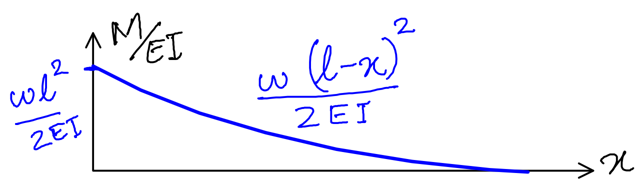






$$M(x) = \frac{w(l-x)^2}{2}$$

$$V_1 = \int_0^l \frac{w(l-x)^2}{2EI}\,dx = \boxed{\frac{wl^3}{6EI} = \theta(l)}$$



$$M_1 = \int_0^l \frac{w(l-x)^3}{2EI}\,.\,dx = \boxed{\frac{wl^4}{8EI} = y(l)}$$

Boundary and Interior conditions for Conjugate Beams

REAL	CONJUGATE
	$w_c(x) = \frac{M}{EI}(x)$
$\theta(x)$ ⟷	$V_c(x)$
$y(x)$ ⟷	$M_c(x)$

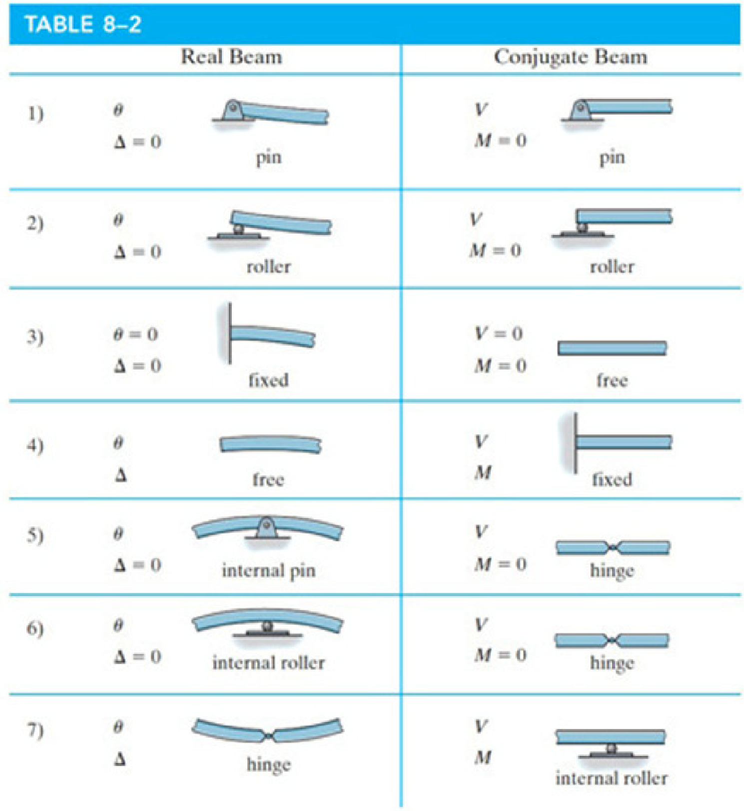


TABLE 8–2

	Real Beam		Conjugate Beam	
1)	θ, $\Delta = 0$	pin	V, $M = 0$	pin
2)	θ, $\Delta = 0$	roller	V, $M = 0$	roller
3)	$\theta = 0$, $\Delta = 0$	fixed	$V = 0$, $M = 0$	free
4)	θ, Δ	free	V, M	fixed
5)	θ, $\Delta = 0$	internal pin	V, $M = 0$	hinge
6)	θ, $\Delta = 0$	internal roller	V, $M = 0$	hinge
7)	θ, Δ	hinge	V, M	internal roller

Figure: tab_08_02

Examples

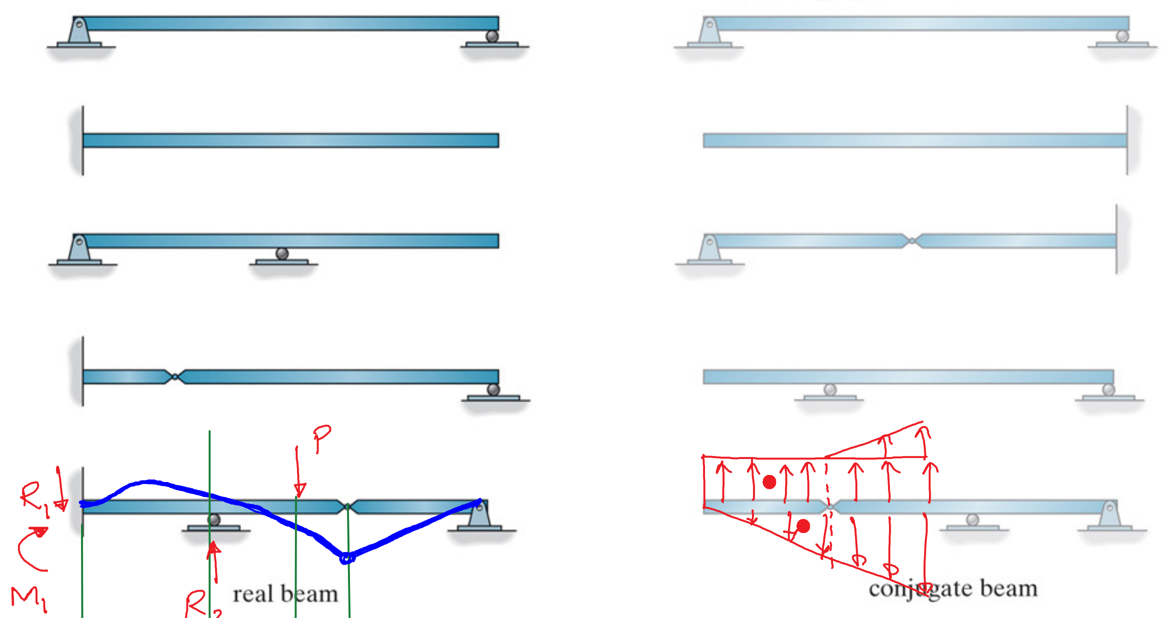
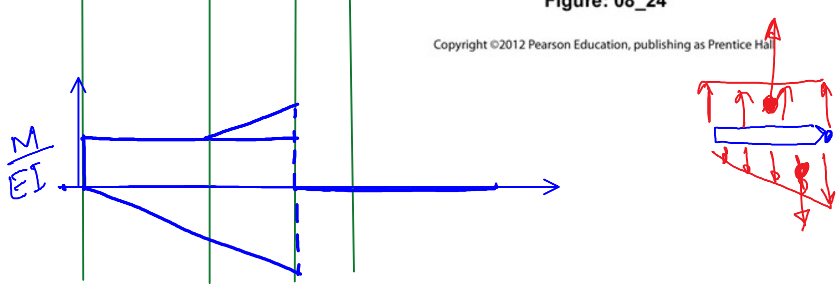


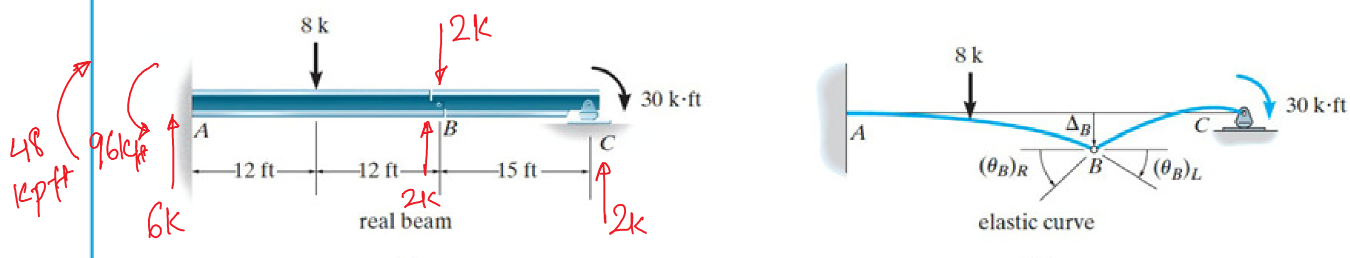
Figure: 08_24



Example

EXAMPLE 8.16

Determine the displacement of the pin at B and the slope of each beam segment connected to the pin for the compound beam shown in Fig. 8–28*a*. $E = 29(10^3)$ ksi, $I = 30$ in^4.

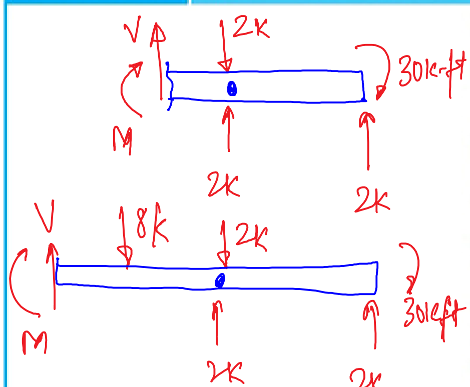


real beam (a)

elastic curve (b)

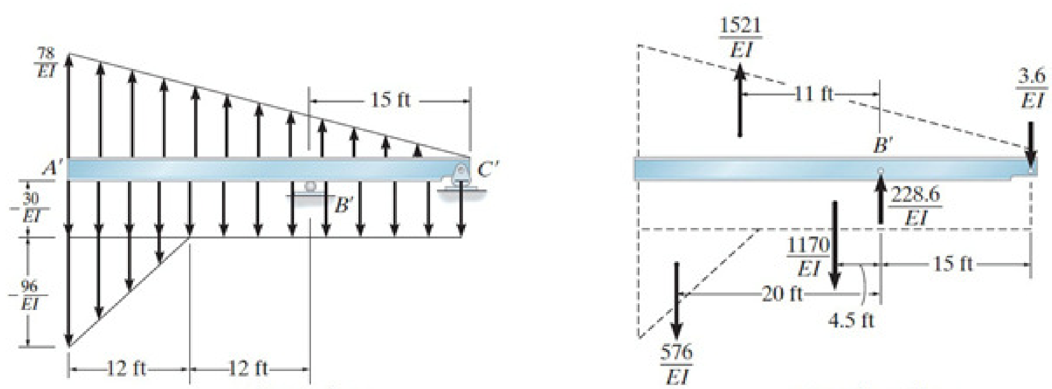
Fig. 8–28

EXAMPLE 8.16 CONTINUED



SOLUTION

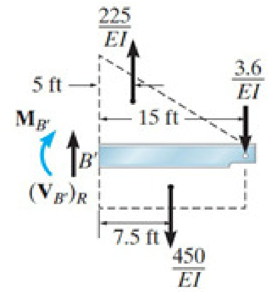
Conjugate Beam. The elastic curve for the beam is shown in Fig. 8–28*b* in order to identify the unknown displacement Δ_B and the slopes $(\theta_B)_L$ and $(\theta_B)_R$ to the left and right of the pin. Using Table 8–2, the conjugate beam is shown in Fig. 8–28*c*. For simplicity in calculation, the *M/EI* diagram has been drawn in *parts* using the principle of superposition as described in Sec. 4–5. In this regard, the real beam is thought of as cantilevered from the left support, *A*. The moment diagrams for the 8-k load, the reactive force $C_y = 2$ k, and the 30-k · ft loading are given. Notice that negative regions of this diagram develop a downward distributed load and positive regions have a distributed load that acts upward.



conjugate beam (c)

external reactions (d)

Equilibrium. The external reactions at B' and C' are calculated first and the results are indicated in Fig. 8–28*d*. In order to determine $(\theta_B)_R$, the conjugate beam is sectioned just to the *right* of B' and the shear force $(V_{B'})_R$ is computed, Fig. 8–28*e*. Thus,



(e)

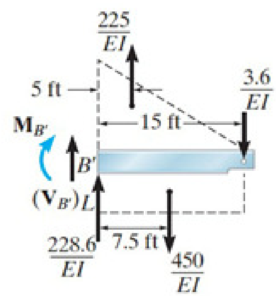
$$+\uparrow \Sigma F_y = 0; \qquad (V_{B'})_R + \frac{225}{EI} - \frac{450}{EI} - \frac{3.6}{EI} = 0$$

$$(\theta_B)_R = (V_{B'})_R = \frac{228.6 \text{ k} \cdot \text{ft}^2}{EI}$$

$$= \frac{228.6 \text{ k} \cdot \text{ft}^2}{[29(10^3)(144) \text{ k/ft}^2][30/(12)^4] \text{ ft}^4}$$

$$= 0.0378 \text{ rad} \qquad \textit{Ans.}$$

The internal moment at B' yields the displacement of the pin. Thus,



(f)

$$\circlearrowleft + \Sigma M_{B'} = 0; \qquad -M_{B'} + \frac{225}{EI}(5) - \frac{450}{EI}(7.5) - \frac{3.6}{EI}(15) = 0$$

$$\Delta_B = M_{B'} = -\frac{2304 \text{ k} \cdot \text{ft}^3}{EI}$$

EXAMPLE 8.16 CONTINUED

$$= \frac{-2304 \text{ k} \cdot \text{ft}^3}{[29(10^3)(144) \text{ k/ft}^2][30/(12)^4] \text{ ft}^4}$$

$$= -0.381 \text{ ft} = -4.58 \text{ in.} \qquad \textit{Ans.}$$

The slope $(\theta_B)_L$ can be found from a section of beam just to the *left* of B', Fig. 8–28*f*. Thus,

$$+\uparrow \Sigma F_y = 0; \qquad (V_{B'})_L + \frac{228.6}{EI} + \frac{225}{EI} - \frac{450}{EI} - \frac{3.6}{EI} = 0$$

$$(\theta_B)_L = (V_{B'})_L = 0 \qquad \textit{Ans.}$$

Obviously, $\Delta_B = M_{B'}$ for this segment is the *same* as previously calculated, since the moment arms are only slightly different in Figs. 8–28*e* and 8–28*f*.

Example:

Determine the slope and deflection at C using the conjugate beam analogy.

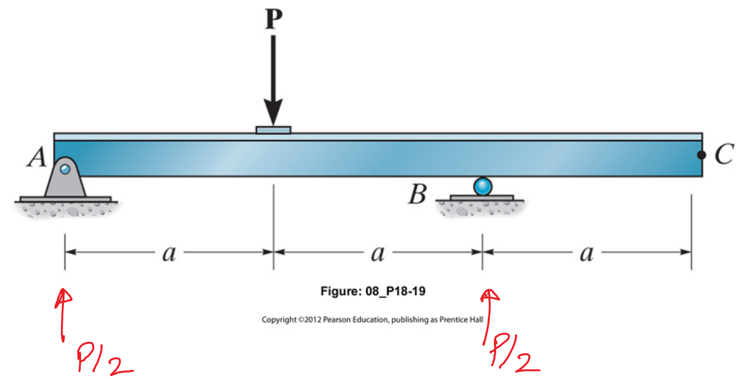
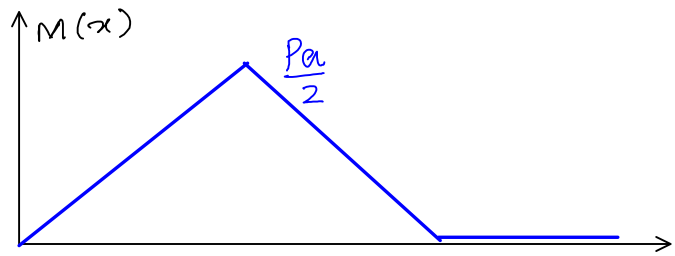


Figure: 08_P18-19

Reactions : $P/2$ @ $0,\ 2a$

Bending moment :

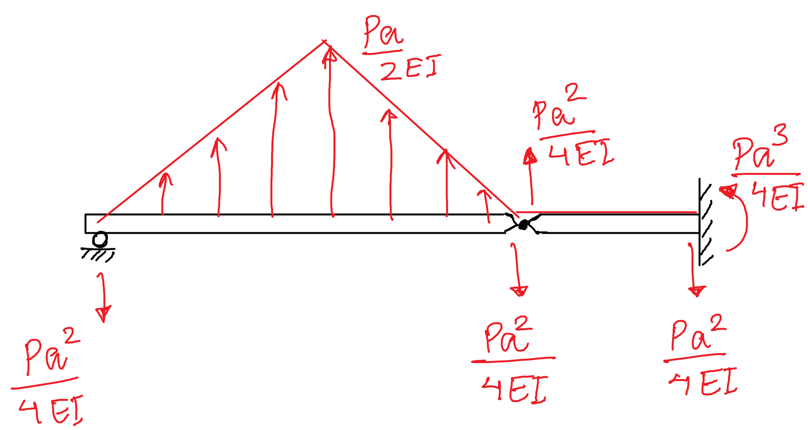
$$M(x) = \begin{cases} \frac{P}{2}x & 0<x<a \\ \frac{P}{2}(2a-x) & a<x<2a \end{cases}$$



Conjugate Beam :

$\theta \leftarrow V_C$

$y \leftarrow M_C$



At C :

$$\theta(x_c) = V_c = \frac{Pa^2}{4EI}$$

$$y(x_c) = M_c = \frac{Pa^3}{4EI}$$