CE-570 Advanced Structural Mechanics

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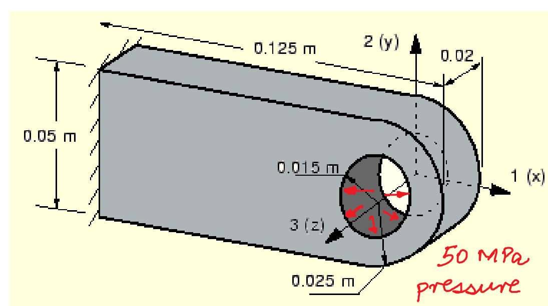
The BIG Picture

- What is Mechanics?
  - Mechanics is study of *how things work*: how anything works, how the world works!
    People ask: "Do you understand the mechanics of _____?" It could be:
    Do you understand the mechanics of this building/bridge? (How loads are being carried?)
    Do you understand the mechanics of heat transfer?
    Do you understand the mechanics of how this automobile works?
    Do you understand the mechanics of flight?
    Do you understand fracture mechanics / wave mechanics / geo-mechanics / thermo-mechanics / electro-mechanics / celestial mechanics / quantum mechanics etc. etc.
    In that sense, mechanics is almost synonymous with Physics. However, mechanics is really a branch of Physics.
  - Within the context of Civil Engineering and Structural Engineering, mechanics is typically used to mean:
    - Rigid body mechanics (Statics / Dynamics),
    - Mechanics of (deformable) materials,
    - Continuum mechanics (including solid / fluid mechanics),
    - Structural mechanics

    (typical courses in undergraduate / graduate curricula: all based on Newtonian Mechanics).

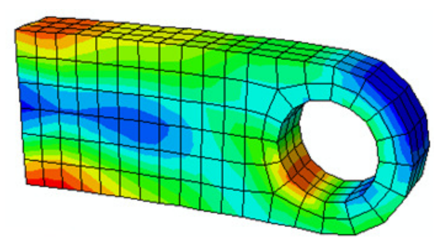
- Continuum mechanics
  - Study of the behavior of continuous bodies (solid/fluid).
    Note: we assume that the "macro-scale" behavior of continuous bodies is not affected by the "micro-scale" (atomic/molecular) structure of their constituent materials.
  - Example Problem statement:



Given:
body / geometry,
boundary conditions,
material properties,
loads

Find:
Solution (displacements, strains, stresses etc.) everywhere in the body.

$\underline{u}(\underline{x})$ $\underset{\sim}{E}(\underline{x})$ $\underset{\sim}{S}(\underline{x})$



Using: Governing partial differential equations (PDEs)

(+BCs)

$$\mathrm{div}(\underset{\sim}{S}) + \underline{b} = \rho\ddot{\underline{u}}$$

$$\underset{\sim}{E} = \tfrac{1}{2}(\nabla\underline{u} + \nabla\underline{u}^T)$$

$$\underset{\sim}{S} = \lambda\, tr(\underset{\sim}{E})\,\underset{\sim}{I} + 2\mu\,\underset{\sim}{E}$$

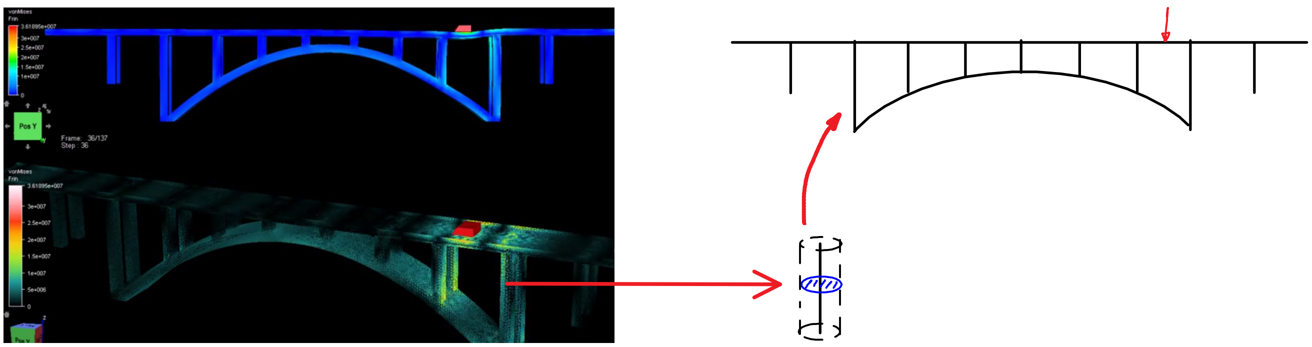
(for small strain linear elasticity, for example)

In addition, if anything is changing with time, then find everything at all times of interest!

$\underline{u}(\underline{x},t)$; $\underset{\sim}{E}(\underline{x},t)$; $\underset{\sim}{S}(\underline{x},t)$

- Structural Mechanics
    - Builds upon continuum (solid) mechanics
    - Make assumptions regarding the displacement field within individual structural members
    - Reduce the "number" of unknowns (dimensionality of the problem)

  Examples: Beam theory, plate theory, shell theory

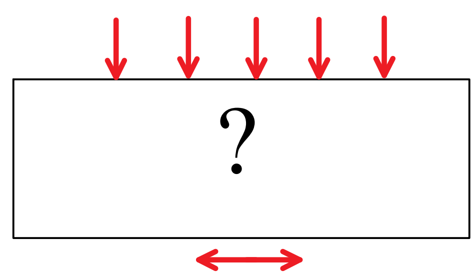


http://youtu.be/IgfWjCvCSuk

- Role of approximate numerical solutions
    - Analytical (exact) solutions to the governing PDEs are not possible in general.
    - One can obtain good approximate solutions, using Finite Element Method (FEM) for example.
    - Understanding the underlying mechanics and solution methods is very important to appreciate limitations of approximate solutions and interpret numerical results correctly.

- Structural Mechanics in relation to Structural Analysis and Design
    - Structural Analysis consists of techniques to solve problems in Structural Mechanics (primarily for beam and frame structures, and use a lot of (conservative) approximations)
        - Determinate structures: Find reactions, internal forces, and then displacements
        - Indeterminate structures: Force/flexibility method; Displacement/Stiffness methods
        - Structural dynamics: Study of structures subject to dynamic loads.
    - Structural Design is an inverse problem:

      Given:
      All possible loads (combinations)
      Permissible displacements, strains, stresses
      Find: A structure that fulfills these constraints!
      (i.e. Geometry, Boundary conditions, materials etc.)
      Approach:
      Assume a solution;
      Check with Structural Analysis / detailed FEM
      Refine as needed.



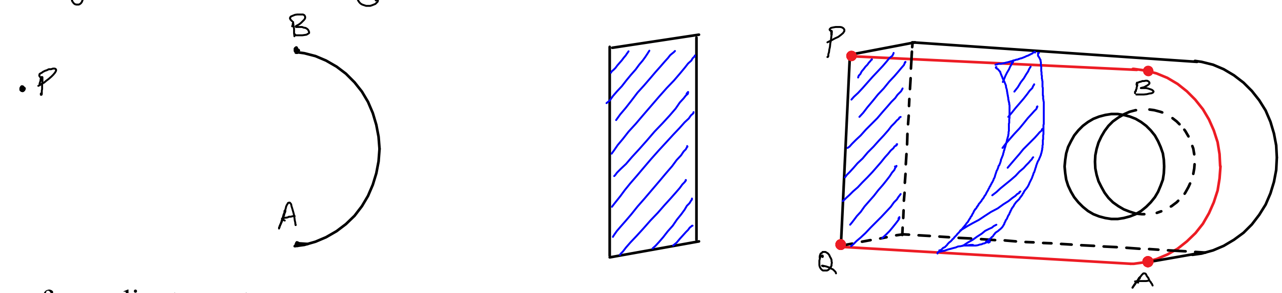
- Objectives of this course
    - Gain in-depth understanding of the basic principles of continuum (solid) mechanics
    - Learn about exact and approximate (numerical) solution methods for governing PDEs
    - Introduction to Variational Principles and concepts in static stability

Chapter 1: Mathematical Preliminaries

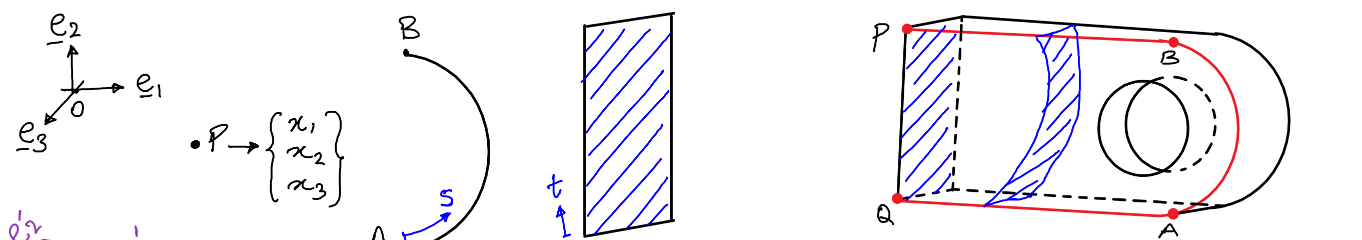
In order to state most problems in mechanics, we need to define some physical entities such continuous bodies, surfaces, curves and points.

Definitions of geometric objects:

Point $\mathcal{P}$ Curve $\mathcal{C}$ Surface $\mathcal{S}$ Body $\mathcal{B}$



Choice of coordinate system:
Location of the Origin and orientation of basis vectors defines a coordinate system.



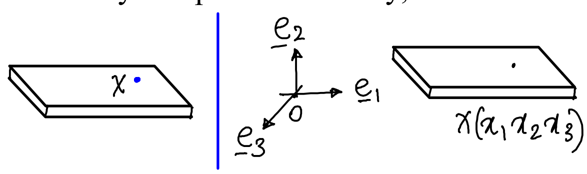
$$\mathcal{P} \to \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

$$\mathcal{C}: s \to \begin{Bmatrix} x_1(s) \\ x_2(s) \\ x_3(s) \end{Bmatrix} \qquad \mathcal{S}: (s,t) \to \begin{Bmatrix} x_1(s,t) \\ x_2(s,t) \\ x_3(s,t) \end{Bmatrix} \qquad \mathcal{B}: (r,s,t) \to \begin{Bmatrix} x_1(r,s,t) \\ x_2(r,s,t) \\ x_3(r,s,t) \end{Bmatrix}$$

We will restrict ourselves to right-handed, orthonormal, Cartesian coordinate systems.

Scalars and scalar fields
Physical quantities with magnitude only. Examples: temperature, density etc.
Denoted with lower case Latin / Greek letters: a, b, c ---- ; α, β, γ, ----

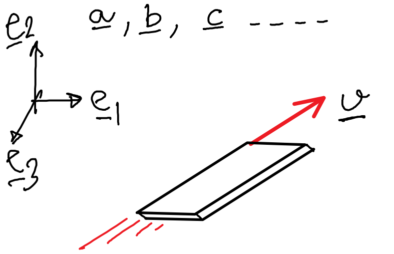
As opposed to "temperature at a point" or "density at a point" in a body, one can also have scalar fields as functions of position:
Example: Temperature field $\theta(X)$
Density field $\rho(X)$



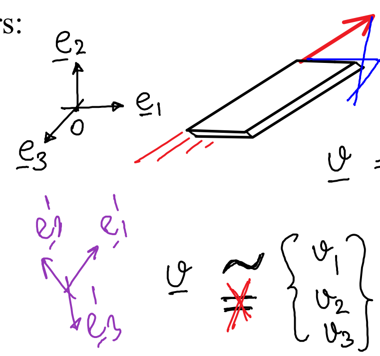
$\theta(x_1, x_2, x_3)$
$\rho(x_1, x_2, x_3)$

Vectors
Physical quantities that need magnitude and direction for defining.
Examples: velocity, force etc.
Denoted with underlined lower case Latin letters:

$\underline{a}, \underline{b}, \underline{c}$ ----



$$\underline{e}_1 \sim \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \underline{e}_2 \sim \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \underline{e}_3 \sim \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$



$$\underline{v} = v_1 \underline{e}_1 + v_2 \underline{e}_2 + v_3 \underline{e}_3 = v_1' \underline{e}_1' + v_2' \underline{e}_2' + v_3' \underline{e}_3'$$

$$\underline{v} = \sum_{i=1}^{3} v_i \underline{e}_i = \sum_{i=1}^{3} v_i' \underline{e}_i'$$

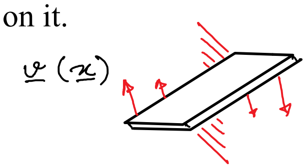
$$\underline{v} \cancel{\sim} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} \sim \begin{Bmatrix} v_1' \\ v_2' \\ v_3' \end{Bmatrix}$$

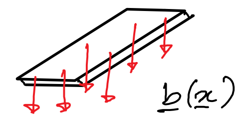
Note: Position "vector" of a point is not strictly a vector since it depends on the definition of a coordinate system. The position "vector" changes if one changes the coordinate system.

Vector fields

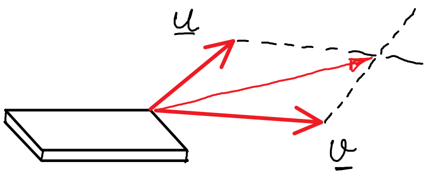
Similar to scalar fields, we can have vector fields as a function of position: each point in a body may have a different velocity or force acting on it.

Example: Velocity field
(Distributed) Force field

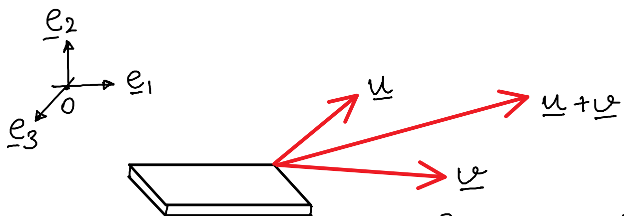




Vector Addition



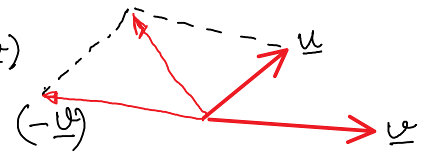
Follows parallelogram law.



$$\underset{\sim}{u}+\underset{\sim}{v} = \sum_{i=1}^{3}(u_i\, \underset{\sim}{e}_i) + \sum_{i=1}^{3} v_i\, \underset{\sim}{e}_i = \sum_{i=1}^{3}(u_i+v_i)\,\underset{\sim}{e}_i$$

- Subtraction, Additive Inverse

$$\underset{\sim}{u}-\underset{\sim}{v} = \underset{\sim}{u}+(-\underset{\sim}{v})$$
$$\underset{\sim}{u}+(-\underset{\sim}{u}) = \underset{\sim}{0}$$



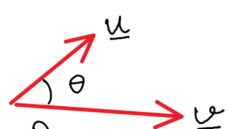
- Properties of vector addition
  Commutative
  Associative

$$\underset{\sim}{u}+\underset{\sim}{v} = \underset{\sim}{v}+\underset{\sim}{u}$$
$$\underset{\sim}{u}+(\underset{\sim}{v}+\underset{\sim}{w}) = (\underset{\sim}{u}+\underset{\sim}{v})+\underset{\sim}{w}$$

Vector Products

- Scalar multiplication (scales the length of the vector)

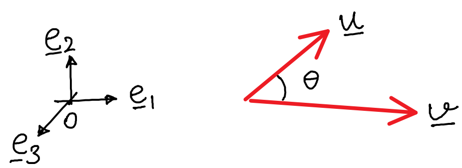
- Dot product



$$\underset{\sim}{u}\cdot\underset{\sim}{v} = \|\underset{\sim}{u}\|\,\|\underset{\sim}{v}\|\cos\theta$$
$$= \tfrac{1}{2}\left(\|\underset{\sim}{u}\|^2+\|\underset{\sim}{v}\|^2-\|\underset{\sim}{v}-\underset{\sim}{u}\|^2\right)$$

where

$$\|\underset{\sim}{u}\|^2 = \underset{\sim}{u}\cdot\underset{\sim}{u}$$



$$\underset{\sim}{u}\cdot\underset{\sim}{v} = \left(\sum_{i=1}^{3} u_i\, \underset{\sim}{e}_i\right)\cdot\left(\sum_{j=1}^{3} v_j\, \underset{\sim}{e}_j\right)$$
$$= \sum_{i=1}^{3}\sum_{j=1}^{3} u_i v_j\,(\underset{\sim}{e}_i\cdot\underset{\sim}{e}_j)$$
$$= \sum_{i=1}^{3}\sum_{j=1}^{3} u_i \cancel{v_j}\, \delta_{i\cancel{j}}$$

contraction

$$= \sum_{i=1}^{3} u_i v_i = u_1v_1+u_2v_2+u_3v_3$$

Note:
Kronecker Delta:

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = \delta_{ij} = \underset{\sim}{e}_i\cdot\underset{\sim}{e}_j = \begin{cases}0 & \text{if } i\neq j\\ 1 & \text{if } i=j\end{cases}$$

For example:

$$\sum_{i=1}^{3}\sum_{j=1}^{3} u_i v_j\,(\underset{\sim}{e}_i\cdot\underset{\sim}{e}_j) = u_1v_1\overset{\delta_{11}=1}{\boxed{(\underset{\sim}{e}_1\cdot\underset{\sim}{e}_1)}} + u_1v_2\cancelto{0}{(\underset{\sim}{e}_1\cdot\underset{\sim}{e}_2)} + u_1v_3\cancelto{0}{(\underset{\sim}{e}_1\cdot\underset{\sim}{e}_3)} +$$
$$u_2v_1\cancelto{0}{(\underset{\sim}{e}_2\cdot\underset{\sim}{e}_1)} + u_2v_2\overset{\delta_{22}=1}{\boxed{(\underset{\sim}{e}_2\cdot\underset{\sim}{e}_2)}} + u_2v_3\cancelto{0}{(\underset{\sim}{e}_2\cdot\underset{\sim}{e}_3)} +$$
$$u_3v_1\cancelto{0}{(\underset{\sim}{e}_3\cdot\underset{\sim}{e}_1)} + u_3v_2\cancelto{0}{(\underset{\sim}{e}_3\cdot\underset{\sim}{e}_2)} + u_3v_3\overset{\delta_{33}=1}{\boxed{(\underset{\sim}{e}_3\cdot\underset{\sim}{e}_3)}}$$

- <u>Indicial notation and summation convention</u>

  In manipulating the component form of complicated vector expressions, we can utilize some shortcuts:
  - Write the expression in terms of *free* and *repeated / dummy* indices (occurring exactly twice),
  - Omit the summation sign (assuming that **<u>summation is implied</u>** for *repeated* / dummy indices),
  - Make use of the Kronecker delta **<u>contraction</u>** property.

<u>Examples</u>:

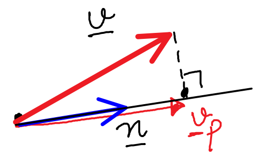
→ Vector $\underline{v} = v_1 \underline{e}_1 + v_2 \underline{e}_2 + v_3 \underline{e}_3 = \sum_{i=1}^{3} v_i \underline{e}_i \equiv v_i \underline{e}_i$

implied repeated

→ Projection of a vector onto another unit vector:

$$\underline{v}_p = (\underline{v} \cdot \underline{n})\, \underline{n} = \left[\sum_{i=1}^{3} v_i n_i\right] \underline{n} = (v_i n_i)\, \underline{n}$$

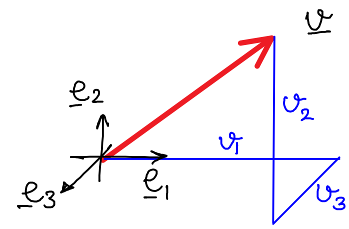
implied repeated



→ Components of a vector:

$$\underline{v} \cdot \underline{e}_i = \left(\sum_{j=1}^{3} v_j \underline{e}_j\right) \cdot \underline{e}_i$$

free index, implied, repeated, free index



$$= v_j (\underline{e}_j \cdot \underline{e}_i) = \cancel{v_j}\, \delta_{ji} = v_i$$

$$\boxed{\underline{v} \cdot \underline{e}_i = v_i} \longrightarrow 3\begin{cases} \underline{v} \cdot \underline{e}_1 = v_1 & (i=1) \\ \underline{v} \cdot \underline{e}_2 = v_2 & (i=2) \\ \underline{v} \cdot \underline{e}_3 = v_3 & (i=3) \end{cases}$$

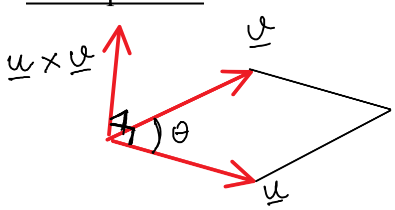
→ Norm: $\|\underline{u}\| = \sqrt{\underline{u} \cdot \underline{u}} = \left[\sum_{i=1}^{3} u_i u_i\right]^{1/2} = \sqrt{u_i u_i}$

implied repeated

$$= \sqrt{u_1 u_1 + u_2 u_2 + u_3 u_3}$$

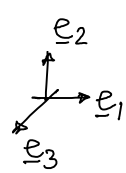
→ Note: $\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = \boxed{3}$

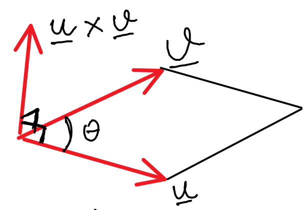
- <u>Cross product</u>



$\|\underline{u} \times \underline{v}\| = \|\underline{u}\|\, \|\underline{v}\| \sin\theta$

(Area of parallelogram)





$$\underline{u} \times \underline{v} = \left(\sum_{i=1}^{3} u_i \underline{e}_i\right) \times \left(\sum_{j=1}^{3} v_j \underline{e}_j\right) = u_i v_j (\underline{e}_i \times \underline{e}_j)$$

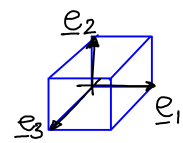
$$= u_1 v_1 \underbrace{\cancel{(\underline{e}_1 \times \underline{e}_1)}}_{0} + u_1 v_2 \underbrace{(\underline{e}_1 \times \underline{e}_2)}_{+\underline{e}_3} + u_1 v_3 \underbrace{(\underline{e}_1 \times \underline{e}_3)}_{-\underline{e}_2}$$

$$+ u_2 v_1 \underbrace{(\underline{e}_2 \times \underline{e}_1)}_{-\underline{e}_3} + u_2 v_2 \underbrace{\cancel{(\underline{e}_2 \times \underline{e}_2)}}_{0} + u_2 v_3 \underbrace{(\underline{e}_2 \times \underline{e}_3)}_{+\underline{e}_1}$$

$$+ u_3 v_1 \underbrace{(\underline{e}_3 \times \underline{e}_1)}_{+\underline{e}_2} + u_3 v_2 \underbrace{(\underline{e}_3 \times \underline{e}_2)}_{-\underline{e}_1} + u_3 v_3 \underbrace{\cancel{(\underline{e}_3 \times \underline{e}_3)}}_{0}$$

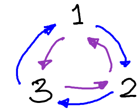
- <u>Permutation (or alternating or Levi-Civita) symbol</u>

Let $\boxed{\epsilon_{ijk} = (\underline{e}_i \times \underline{e}_j)\cdot \underline{e}_k}$



Note :

$\rightarrow$ $$\epsilon_{ijk} = \begin{cases} 0 & \text{if } i=j \text{ or } j=k \text{ or } k=i \\ +1 & \text{if } (i,j,k) \text{ are cyclic : } 123,\ 231,\ 312 \\ -1 & \text{if } (i,j,k) \text{ are acyclic : } 132,\ 321,\ 213 \end{cases}$$



$\rightarrow$ $$\epsilon_{ijk}\,\underline{e}_k = \sum_{k=1}^{3} \epsilon_{ijk}\,\underline{e}_k = [(\underline{e}_i \times \underline{e}_j)\cdot \underline{e}_k]\,\underline{e}_k = (\underline{e}_i \times \underline{e}_j)$$

$$= (\epsilon_{ij1}\,\underline{e}_1 + \epsilon_{ij2}\,\underline{e}_2 + \epsilon_{ij3}\,\underline{e}_3)$$

$$= \underbrace{[\overbrace{(\underline{e}_i \times \underline{e}_j)}^{\underline{v}}\cdot \underline{e}_1]}_{\text{Component of } (\underline{e}_i \times \underline{e}_j) \text{ in } \underline{e}_1}\underline{e}_1 + \underbrace{[\overbrace{(\underline{e}_i \times \underline{e}_j)}^{\underline{v}}\cdot \underline{e}_2]}_{\text{in } \underline{e}_2}\underline{e}_2 + \underbrace{[\overbrace{(\underline{e}_i \times \underline{e}_j)}^{\underline{v}}\cdot \underline{e}_3]}_{\text{in } \underline{e}_3}\underline{e}_3$$

$$\boxed{\epsilon_{ijk}\,\underline{e}_k = (\underline{e}_i \times \underline{e}_j)}$$

Thus cross product :

$$\boxed{\underline{u}\times\underline{v} = u_i v_j (\underline{e}_i \times \underline{e}_j) = \epsilon_{ijk}\, u_i\, v_j\, \underline{e}_k}$$

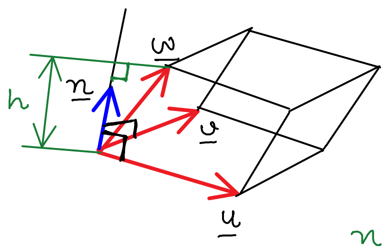
(Triple Sum)

Note : $\rightarrow$ $\underline{u}\times\underline{v} = -\underline{v}\times\underline{u}$

$\rightarrow$ $\alpha\underline{u}\times(\beta\underline{v} + \gamma\underline{w}) = \alpha\beta(\underline{u}\times\underline{v}) + \alpha\gamma(\underline{u}\times\underline{w})$

- <u>Scalar Triple product of 3 vectors</u>

$\underline{n} = \dfrac{\underline{u}\times\underline{v}}{\|\underline{u}\times\underline{v}\|}$



$$(\underline{u}\times\underline{v})\cdot\underline{w} = \underbrace{\|\underline{u}\times\underline{v}\|}\ \underbrace{\left(\underline{w}\cdot \overbrace{\frac{(\underline{u}\times\underline{v})}{\|\underline{u}\times\underline{v}\|}}^{\underline{n}}\right)}$$

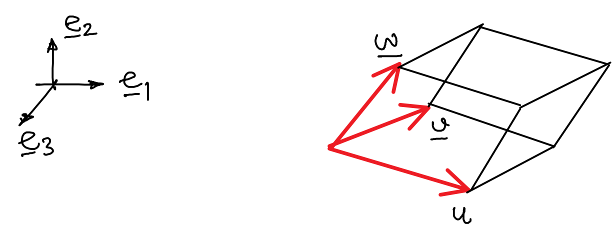
Volume = Area $\times$ h

Note

$\rightarrow$ $(\underline{u}\times\underline{v})\cdot\underline{w}$

$= (\underline{w}\times\underline{u})\cdot\underline{v} = (\underline{v}\times\underline{w})\cdot\underline{u}$

$= -(\underline{u}\times\underline{w})\cdot\underline{v} = -(\underline{w}\times\underline{v})\cdot\underline{u}$



$$(\underline{u}\times\underline{v})\cdot\underline{w} = (\epsilon_{ijk}\, u_i\, v_j\, \underline{e}_k)\cdot(w_l\,\underline{e}_l)$$

$$= \epsilon_{ijk}\, u_i\, v_j\, \cancel{w_l}\, \underbrace{(\underline{e}_k\cdot\underline{e}_l)}_{\delta_{k\cancel{l}}}$$

$$\boxed{(\underline{u}\times\underline{v})\cdot\underline{w} = \epsilon_{ijk}\, u_i\, v_j\, w_k}$$

$$= (u_1 v_2 - u_2 v_1)\, w_3 + (u_2 v_3 - u_3 v_2)\, w_1 + (u_3 v_1 - u_1 v_3)\, w_2$$

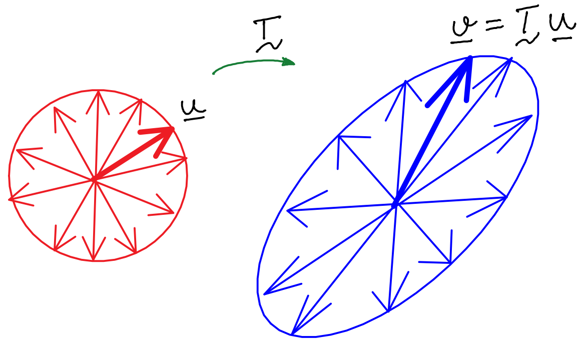
Tensors

In mechanics we often need more general quantities than just scalars and vectors.
Tensors are a generalization of the concept of scalars and vectors

Definition: $\boxed{\text{Tensors are entities that operate upon a vector to produce another vector} \quad \underset{\sim}{T}\ \underset{\sim}{u} = \underset{\sim}{v}}$

Examples: Strain ($\underset{\sim}{E}$) ; Stress ($\underset{\sim}{S}$) ; Identity ($\underset{\sim}{I}$) ;
Moment of Inertia ($\underset{\sim}{I_p}$) ; Projection ($\underset{\sim}{P}$)

A good way to think about tensors is in terms of their effect on an arbitrary vector:

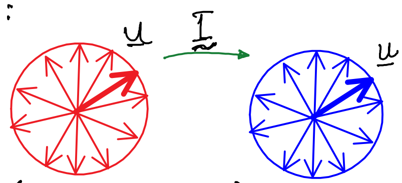


$$\underset{\sim}{T}\ \underset{\sim}{u} = \underset{\sim}{v}$$

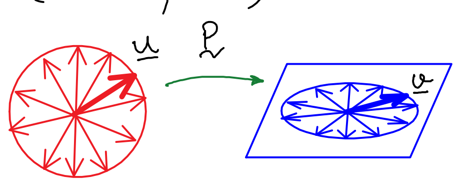
Note: Its effect can cause change in length & direction (visualized as an ellipsoid)

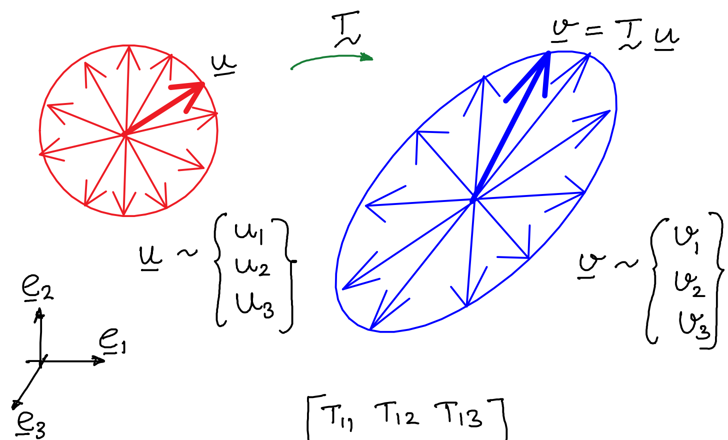
Examples:

Identity:



Projection (on a plane):





$$\underset{\sim}{u} \sim \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \qquad \underset{\sim}{v} \sim \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}$$

$$\underset{\sim}{T} \sim \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$

such that

$$\underset{\sim}{T} \quad \underset{\sim}{u} \quad = \quad \underset{\sim}{v}$$

$$\begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}$$

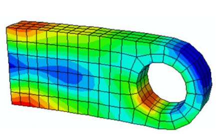
$$T_{ij}\ u_j = v_i$$

i.e.

$$T_{11}\ u_1 + T_{12}\ u_2 + T_{13}\ u_3 = v_1$$
$$T_{21}\ u_1 + T_{22}\ u_2 + T_{23}\ u_3 = v_2$$
$$T_{31}\ u_1 + T_{32}\ u_2 + T_{33}\ u_3 = v_3$$

Tensor fields:

Just as scalar and vector fields, we can have tensor fields *i.e.* tensor as a function of position: $\boldsymbol{T}(\boldsymbol{x})$ **:** $\underset{\sim}{T}(\underset{\sim}{x})$
Note that writing a tensor field as $\boldsymbol{T}(\boldsymbol{x})$ does ***not*** mean that $\boldsymbol{T}$ is operating on $\boldsymbol{x}$.
It means that the $\boldsymbol{T}(\boldsymbol{x})$ is a function of $\boldsymbol{x}$ and still operates on a vector $\boldsymbol{u}(\boldsymbol{x})$ ***at that point***.
Written as: $\boldsymbol{T}(\boldsymbol{x})\ \boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{v}(\boldsymbol{x})$ $\qquad \underset{\sim}{T}(\underset{\sim}{x})\ \underset{\sim}{u}(\underset{\sim}{x}) = \underset{\sim}{v}(\underset{\sim}{x})$
or simply as: $\boldsymbol{T}\ \boldsymbol{u} = \boldsymbol{v}$ $\qquad \underset{\sim}{T}\ \underset{\sim}{u} = \underset{\sim}{v}$

Example:  Stress or strain fields.

Properties of a tensor:

- $\underset{\sim}{T}$ is a linear operator $\Rightarrow \underset{\sim}{T}(\alpha \underline{u} + \beta \underline{v}) = \alpha \underset{\sim}{T}\underline{u} + \beta \underset{\sim}{T}\underline{v}$
- Tensors can be added/subtracted $\Rightarrow (\underset{\sim}{T} + \underset{\sim}{S})\underline{u} = \underset{\sim}{T}\underline{u} + \underset{\sim}{S}\underline{u}$
- Scalar multiplication $\Rightarrow (\alpha \underset{\sim}{T})\underline{u} = \alpha(\underset{\sim}{T}\underline{u})$

Tensor product of two vectors

It is possible to construct a tensor from two vectors by using a special operation called a tensor product:

Let $\underset{\sim}{T} = \underline{u} \otimes \underline{v}$

such that $\underset{\sim}{T}\underline{w} = (\underline{u} \otimes \underline{v})\underline{w} \equiv \underline{u}(\underline{v} \cdot \underline{w})$

$\otimes$ is also called dyadic product.

In matrix notation: $\underset{\sim}{T} = \underline{u} \otimes \underline{v}$

such that: $(\underline{u} \otimes \underline{v})\underline{w} = \underline{u}(\underline{v} \cdot \underline{w})$

$$\underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}}_{\text{(outer product)}\,[\underset{\sim}{T}]} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} v_i w_i$$

Just like any vector can be expressed in terms of basis vectors: $\underline{e}_1, \underline{e}_2, \underline{e}_3$: $\underline{u} = u_i \underline{e}_i$

We can also construct basis tensors: $(\underline{e}_i \otimes \underline{e}_j)$ such that $(\underline{e}_i \otimes \underline{e}_j)\underline{e}_k = \underline{e}_i(\underline{e}_j \cdot \underline{e}_k) = \underline{e}_i \delta_{jk}$

Using this: $\underset{\sim}{T} = \sum_{i=1}^{3} \sum_{j=1}^{3} T_{ij} (\underline{e}_i \otimes \underline{e}_j)$

$$= \left.\begin{matrix} T_{11}(\underline{e}_1 \otimes \underline{e}_1) + T_{12}(\underline{e}_1 \otimes \underline{e}_2) + T_{13}(\underline{e}_1 \otimes \underline{e}_3) \\ + T_{21}(\underline{e}_2 \otimes \underline{e}_1) + T_{22}(\underline{e}_2 \otimes \underline{e}_2) + T_{23}(\underline{e}_2 \otimes \underline{e}_3) \\ + T_{31}(\underline{e}_3 \otimes \underline{e}_1) + T_{32}(\underline{e}_3 \otimes \underline{e}_2) + T_{33}(\underline{e}_3 \otimes \underline{e}_3) \end{matrix}\right\} \sim \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$

$$\underline{e}_1 \otimes \underline{e}_1 \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

such that $\underset{\sim}{T}\underline{u} = \left(\sum_{i=1}^{3}\sum_{j=1}^{3} T_{ij}\, \underline{e}_i \otimes \underline{e}_j\right)\left(\sum_{k=1}^{3} u_k \underline{e}_k\right) = T_{ij}\, u_{\cancel{k}}\, \underline{e}_i \underbrace{(\underline{e}_j \cdot \underline{e}_k)}_{\delta_{j\cancel{k}}}$

$= T_{ij} u_j \underline{e}_i$

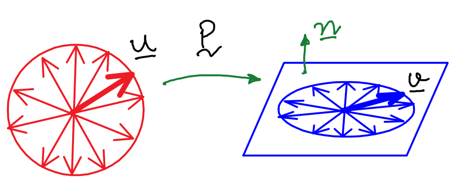
$$\underline{v} = \underset{\sim}{T}\underline{u} = \underbrace{(T_{ij} u_j)}\underline{e}_i = \underbrace{v_i}\underline{e}_i$$

$T_{i\cancel{j}}\, u_k\, \underline{e}_i\, \delta_{\cancel{j}k}$

$= T_{ik}\, u_k\, \underline{e}_i$

Examples:

- Identity $\underset{\sim}{I} = \delta_{ij}(\underline{e}_i \otimes \underline{e}_j) \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- Projection (on a plane): $\underset{\sim}{P} = \underset{\sim}{I} - \underline{n} \otimes \underline{n}$



such that $\underbrace{\underset{\sim}{P}\underline{u}} = \underset{\sim}{I}\underline{u} - (\underline{n} \otimes \underline{n})\underline{u}$

$\underline{v} = \underline{u} - (\underline{u} \cdot \underline{n})\underline{n}$

- Tensor Composition (product of 2 tensors to get another tensor)

Let $\quad \underset{\sim}{R} = \underset{\sim}{S}\,\underset{\sim}{T}$

$$\underset{\sim}{R}\,\underline{u} = (\underset{\sim}{S}\,\underset{\sim}{T})\,\underline{u} \equiv \underset{\sim}{S}(\underset{\sim}{T}\,\underline{u})$$

Note :

$\rightarrow \quad \underset{\sim}{S}\,\underset{\sim}{T} \neq \underset{\sim}{T}\,\underset{\sim}{S}$

$\rightarrow \quad \underset{\sim}{S}(\underset{\sim}{T}\,\underset{\sim}{v}) = (\underset{\sim}{S}\,\underset{\sim}{T})\,\underset{\sim}{v} = \underset{\sim}{S}\,\underset{\sim}{T}\,\underset{\sim}{v}$

$\rightarrow \quad \underset{\sim}{T}^2 = \underset{\sim}{T}\,\underset{\sim}{T}$

$\rightarrow \quad \underset{\sim}{S}(\underset{\sim}{T} + \underset{\sim}{U}) = \underset{\sim}{S}\,\underset{\sim}{T} + \underset{\sim}{S}\,\underset{\sim}{U}$

$$\underset{\sim}{S}(\underset{\sim}{T}\,\underline{u}) = S_{ij}(\underline{e}_i \otimes \underbrace{\underline{e}_j)\overbrace{T_{kl}\,u_l\,\underline{e}_k}^{\underset{\sim}{T}\,\underline{u}}}_{\delta_{jk}}$$

$$= S_{ij}\,T_{\cancel{k}l}\,u_l\,\underline{e}_i \qquad (k \to j)$$

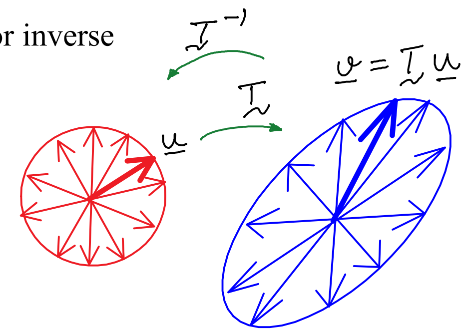
$$= \underbrace{(S_{ij}\,T_{jl})}_{\text{matrix multiplication of } [S][T]}\,u_l\,\underline{e}_i$$

Thus $\quad \underset{\sim}{S}\,\underset{\sim}{T} = S_{ik}(\underline{e}_i \otimes \overbrace{\underline{e}_k)\,T_{lj}(\underline{e}_l}^{\delta_{kl}} \otimes \underline{e}_j)$

$$= S_{ik}\,T_{\cancel{l}j}\,(\underline{e}_i \otimes \underline{e}_j)\,\delta_{k\cancel{l}}$$

$$= S_{ik}\,T_{kj}\,(\underline{e}_i \otimes \underline{e}_j)$$

- Tensor inverse

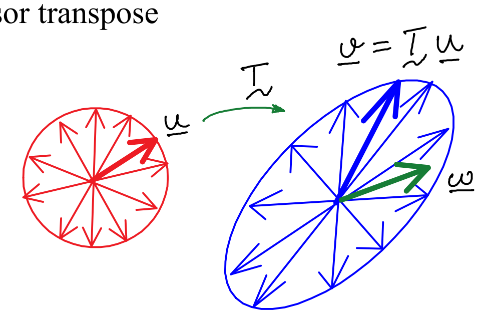


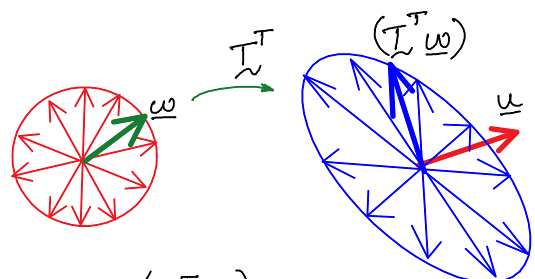
Note : $\quad \underline{v} = \underset{\sim}{T}\,\underline{u}$

$$\underline{u} = \underset{\sim}{T}^{-1}\,\underline{v}$$

$$\underset{\sim}{T}\,\underset{\sim}{T}^{-1} = \underset{\sim}{T}^{-1}\,\underset{\sim}{T} = \underset{\sim}{I}$$

- Tensor transpose





For any 2 arbitrary vectors ***u*** and ***w*** $\qquad \underline{w} \cdot (\underset{\sim}{T}\,\underline{u}) \equiv \underline{u} \cdot (\underset{\sim}{T}^T\,\underline{w})$

Note :

$$w_i\,\underline{e}_i \cdot T_{jk}\,u_k\,\underline{e}_j \quad \longleftrightarrow \quad u_m\,\underline{e}_m \cdot (T^T)_{nl}\,w_l\,\underline{e}_n$$

$$u_k\,w_i\,T_{\cancel{j}k}\,\delta_{i\cancel{j}} \quad \longleftrightarrow \quad u_{\cancel{m}}\,(T^T)_{nl}\,w_l\,\delta_{\cancel{m}n} \quad (m \to n)$$

$$u_k\,w_i\,\boxed{T_{ik}} \quad \longleftrightarrow \quad u_n\,w_l\,(T^T)_{nl}$$

$$u_k\,w_i\,(T^T)_{ki} \qquad \begin{pmatrix} n \to k \\ l \to i \end{pmatrix}$$

$$\Rightarrow \quad T_{ik} = (T^T)_{ki}$$

- Symmetric Tensors

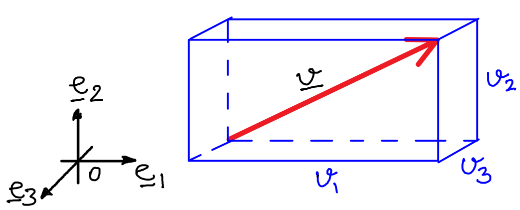
$$\underline{u} \cdot (\underset{\sim}{S}\,\underline{v}) = \underline{v} \cdot (\underset{\sim}{S}\,\underline{u}) \Rightarrow \underset{\sim}{S} = \underset{\sim}{S}^T \qquad (\text{i.e. } S_{ik} = S_{ki})$$

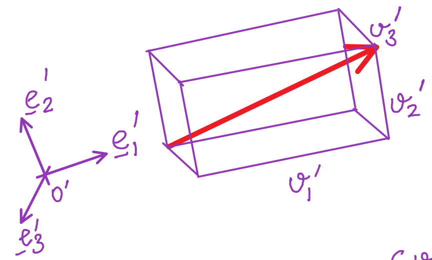
- Skew-symmetric Tensors

$$\underline{u} \cdot (\underset{\sim}{W}\,\underline{v}) = -\,\underline{v} \cdot (\underset{\sim}{W}\,\underline{u}) \Rightarrow \underset{\sim}{W} = -\,\underset{\sim}{W}^T \qquad (\text{i.e. } W_{ik} = -\,W_{ki})$$

Note: Any tensor ***T*** can be expressed as: $\quad \underset{\sim}{T} = \underbrace{\tfrac{1}{2}(\underset{\sim}{T} + \underset{\sim}{T}^T)}_{\text{symmetric}} + \underbrace{\tfrac{1}{2}(\underset{\sim}{T} - \underset{\sim}{T}^T)}_{\text{skew}}$

Change of coordinate system





$$\underset{\sim}{v} = v_i \underset{\sim}{e}_i \sim \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}_{(0,\ \underset{\sim}{e}_1 \underset{\sim}{e}_2 \underset{\sim}{e}_3)} \qquad \underset{\sim}{v} = v_i' \underset{\sim}{e}_i' \sim \begin{Bmatrix} v_1' \\ v_2' \\ v_3' \end{Bmatrix}_{(0',\ \underset{\sim}{e}_1',\ \underset{\sim}{e}_2',\ \underset{\sim}{e}_3')}$$

Note: $\underset{\sim}{v} = v_i \underset{\sim}{e}_i = v_i' \underset{\sim}{e}_i'$

Recall $v_i = \underset{\sim}{v} \cdot \underset{\sim}{e}_i$ and $v_i' = \underset{\sim}{v} \cdot \underset{\sim}{e}_i'$

Thus $v_i = (v_j' \underset{\sim}{e}_j') \cdot \underset{\sim}{e}_i$; $v_i' = (v_j \underset{\sim}{e}_j) \cdot \underset{\sim}{e}_i'$

$$v_i = \underbrace{(\underset{\sim}{e}_j' \cdot \underset{\sim}{e}_i)}_{Q_{ji}} v_j' \quad ; \quad \boxed{Q_{ij} = \underset{\sim}{e}_i' \cdot \underset{\sim}{e}_j} \quad ; \quad v_i' = \underbrace{(\underset{\sim}{e}_i' \cdot \underset{\sim}{e}_j)}_{Q_{ij}} v_j$$

In terms of matrices:

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{21} & Q_{31} \\ Q_{12} & Q_{22} & Q_{32} \\ Q_{13} & Q_{23} & Q_{33} \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \\ v_3' \end{bmatrix} \quad ; \quad \begin{bmatrix} v_1' \\ v_2' \\ v_3' \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

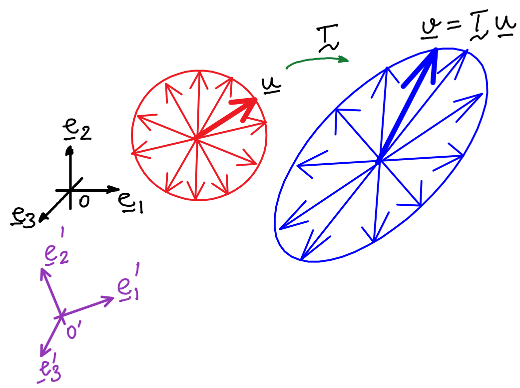
$$[\underset{\sim}{v}] = [Q]^T [\underset{\sim}{v}'] \qquad [\underset{\sim}{v}'] = [\mathbf{Q}][\underset{\sim}{v}]$$

Note: [$\boldsymbol{Q}$] here is a transformation matrix (not a tensor).
[$\boldsymbol{Q}$] is an orthogonal matrix: $[\boldsymbol{Q}][\boldsymbol{Q}]^T = [\boldsymbol{Q}]^T[\boldsymbol{Q}] = [\boldsymbol{I}]$
However a tensor $\boldsymbol{Q}$ can be defined such that $\boldsymbol{e}_i = \boldsymbol{Q}\, \boldsymbol{e}_i'$ (see problem 13 in Hjelmstad)

Transformation of Tensors

$$\underset{\sim}{T} = T_{ij} (\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j) \sim \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}_{(0,\ \underset{\sim}{e}_1 \underset{\sim}{e}_2 \underset{\sim}{e}_3)}$$

$$\underset{\sim}{T} = T_{ij}' (\underset{\sim}{e}_i' \otimes \underset{\sim}{e}_j') \sim \begin{bmatrix} T_{11}' & T_{12}' & T_{13}' \\ T_{21}' & T_{22}' & T_{23}' \\ T_{31}' & T_{32}' & T_{33}' \end{bmatrix}_{(0',\ \underset{\sim}{e}_1' \underset{\sim}{e}_2' \underset{\sim}{e}_3')}$$



Note: Tensor components can be obtained as:

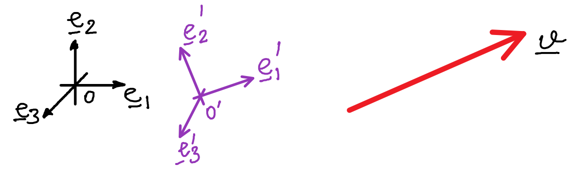
$$T_{ij} = \underset{\sim}{e}_i \cdot (\underset{\sim}{T}\, \underset{\sim}{e}_j) = \underset{\sim}{e}_i \cdot T_{mn} (\underset{\sim}{e}_m \otimes \underset{\sim}{e}_n)\, \underset{\sim}{e}_j = \cancel{T_{mn}}_{ij} \overbrace{(\underset{\sim}{e}_i \cdot \underset{\sim}{e}_m)}^{\delta_{i\cancel{m}}} \overbrace{(\underset{\sim}{e}_n \cdot \underset{\sim}{e}_j)}^{\delta_{\cancel{n}j}} = T_{ij}$$

$$= \underset{\sim}{e}_i \cdot (\underset{\sim}{T}\, \underset{\sim}{e}_j) = \underset{\sim}{e}_i \cdot T_{mn}' (\underset{\sim}{e}_m' \otimes \underset{\sim}{e}_n')\, \underset{\sim}{e}_j = T_{mn}' \underbrace{(\underset{\sim}{e}_n' \cdot \underset{\sim}{e}_j)}_{Q_{nj}} \underbrace{(\underset{\sim}{e}_m' \cdot \underset{\sim}{e}_i)}_{Q_{mi}}$$

$$\boxed{T_{ij} = (Q)^T_{im}\ T_{mn}'\ Q_{nj}} \quad \text{and} \quad \boxed{T_{ij}' = Q_{im}\ T_{mn} (Q)^T_{nj}}$$

Tensor Invariants (Quantities that don't change, no matter which coordinate system is chosen)

Example of an invariant for a vector:



magnitude $\|\underset{\sim}{v}\| = \sqrt{v_i v_i}$

$$\|\underset{\sim}{v}\| = \sqrt{v'_i v'_i} = \sqrt{(Q_{im} v_m)(Q_{in} v_n)} = \sqrt{\delta_{nm} v_{\cancel{m}} v_n}$$

$$(Q_{ni})^T \; Q_{im} \qquad = \sqrt{v_n v_n}$$

Similarly, a tensor invariant is a function of the tensor components (in any coordinate system): $f(T_{ij})$

In a different coordinate system, the tensor components would be given by: $T'_{ij} = Q_{im} T_{mn} (Q)^T_{nj}$

For a function to be invariant: $f(T_{ij}) = f(T'_{ij}) \quad \forall \quad [Q]$ rotation

Simply referred to as: $f(\underset{\sim}{T})$ (for any)

Primary Invariants

$$\rightarrow f_1(\underset{\sim}{T}) \equiv T_{ii} \leftarrow tr(\underset{\sim}{T}) \text{ : trace}$$

$$\rightarrow f_2(\underset{\sim}{T}) \equiv T_{ij} T_{ji} \leftarrow tr(\underset{\sim}{T}\underset{\sim}{T}) = tr(\underset{\sim}{T}^2)$$

$$\rightarrow f_3(\underset{\sim}{T}) \equiv T_{ij} T_{jk} T_{ki} \leftarrow tr(\underset{\sim}{T}^3)$$

$$\vdots$$

$$\rightarrow f_n(\underset{\sim}{T}) \equiv T_{i_1 i_2} T_{i_2 i_3} T_{i_3 i_4} \cdots T_{i_n i_1} \leftarrow tr(\underset{\sim}{T}^n)$$

Note:

$$T'_{ii} = Q_{im} T_{mn} (Q)^T_{ni}$$

$$[Q^T][Q] = [I]$$

$$= \delta_{nm} T_{\cancel{m}n}$$

$$= T_{nn} = T_{ii}$$

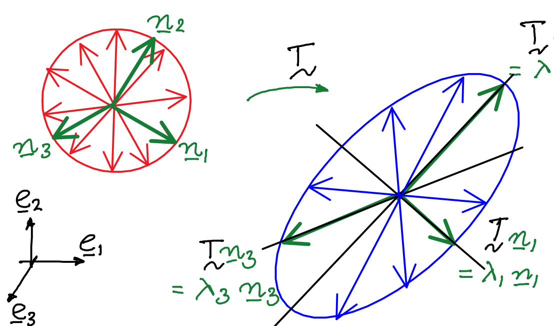
(invariant)

Eigenvalues & Eigenvectors of **Symmetric** Tensors

(**Principal** Invariants)

As mentioned earlier, a tensor operates on a vector to produce another vector (by stretching and/or rotating it). However, for a given symmetric tensor, there are some specific vectors (directions) $\boldsymbol{n}$ on which the action of the tensor is purely stretching (no rotation).

i.e. $\underset{\sim}{T}\,\underset{\sim}{n} = \lambda \underset{\sim}{n}$



Note: $\boldsymbol{n}_i$ are orthogonal

The problem of finding λ and $\boldsymbol{n}$ for a given (symmetric) tensor is called the Eigenvalue problem.

To obtain non-trivial solutions ($\boldsymbol{n} \neq \boldsymbol{0}$):

$$(\underset{\sim}{T} - \lambda \underset{\sim}{I})\underset{\sim}{n} = \underset{\sim}{0}$$

$$\Rightarrow \det(\underset{\sim}{T} - \lambda \underset{\sim}{I}) = 0 \quad \Rightarrow \det \begin{bmatrix} T_{11}-\lambda & T_{12} & T_{13} \\ T_{21} & T_{22}-\lambda & T_{23} \\ T_{31} & T_{32} & T_{33}-\lambda \end{bmatrix} = 0$$

This results in a cubic equation for λ called the characteristic equation:

$$-\lambda^3 + I_T \lambda^2 - II_T \lambda + III_T = 0$$

where I_T, II_T, III_T are called the Principal Invariants of $\boldsymbol{T}$:

$$I_T = tr(\underset{\sim}{T}) = T_{ii}$$

$$II_T = \tfrac{1}{2}\left[(tr(\underset{\sim}{T}))^2 - tr(\underset{\sim}{T}^2)\right] = \tfrac{1}{2}\left[(T_{ii})^2 - (T_{ij}T_{ji})\right]$$

$$III_T = \det(\underset{\sim}{T}) = \tfrac{1}{6}\,\epsilon_{ijk}\,\epsilon_{lmn}\,T_{il}\,T_{jm}\,T_{kn}$$

Solving the characteristic equation:

The cubic polynomial equation, in general, will have 3 roots (Eigenvalues): λ_1, λ_2, λ_3
and 3 corresponding Eigenvectors: $\boldsymbol{n}_1$, $\boldsymbol{n}_2$, $\boldsymbol{n}_3$

Example:

- By hand (factorizing):

$$\underset{\sim}{T} \sim \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & -1 \\ 0 & -1 & 5 \end{bmatrix} \Rightarrow (3-\lambda)\left[(5-\lambda)(5-\lambda) - 1\right] = 0$$

$$\Rightarrow (3-\lambda)\left[\lambda^2 - 10\lambda + 24\right] = 0$$

characteristic equation : $-\lambda^3 + \underbrace{13}_{I_T}\lambda^2 - \underbrace{54}_{II_T}\lambda + \underbrace{72}_{III_T} = 0$

Factorize : $(3-\lambda)(\lambda-6)(\lambda-4) = 0 \Rightarrow \lambda_1 = 3,\ \lambda_2 = 4,\ \lambda_3 = 6$

Corresponding Eigenvectors :

For $\lambda_1 = 3$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} \cancel{v_1}\ 1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \underset{\sim}{n}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

For $\lambda_2 = 4$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ \cancel{v_2}\ 1 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} v_1 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \underset{\sim}{n}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

For $\lambda_3 = 6$

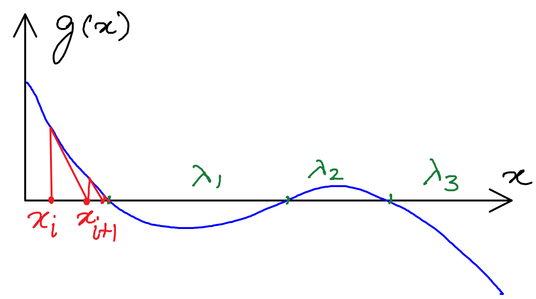
$$\begin{bmatrix} -3 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ \cancel{v_2}\ 1 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} v_1 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \Rightarrow \underset{\sim}{n}_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

- Using numerical non-linear equation solver (Newton's method)

To solve $g(x) = 0$

Expand $\underbrace{g(x_i + \Delta x)}_{0} \cong g(x_i) + \left[\frac{dg}{dx}\right]\Delta x$

$$\Rightarrow \Delta x = \frac{-g(x_i)}{g'(x_i)} \; ; \; x_{i+1} = x_i + \Delta x$$



- Using existing software programs such as MATLAB:

```
>> T = [3 0 0 ; 0 5 -1 ; 0 -1 5]
T =
     3     0     0
     0     5    -1
     0    -1     5
>> [v,d] = eig(T)
v =     n1          n2          n3
    1.0000           0           0
         0     -0.7071     -0.7071
         0     -0.7071      0.7071
d =
     3     0     0      λ1
     0     4     0      λ2
     0     0     6      λ3
```

Note: The Principal invariants I_T, II_T, III_T of $\boldsymbol{T}$ are the coefficients of the characteristic equation:

$I_T =$
```
>> trace(T)
ans =
    13
```

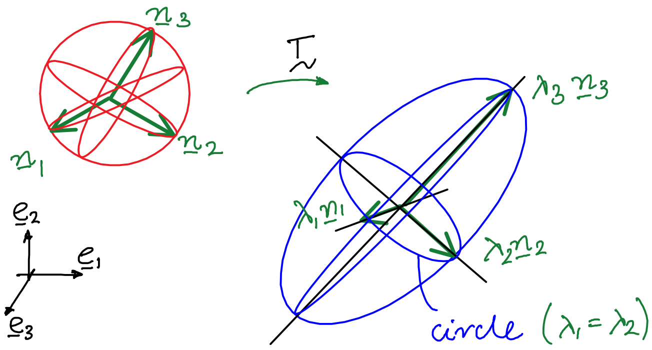
$II_T =$
```
>> 1/2*((trace(T))^2-trace(T^2))
ans =
    54
```

$III_T =$
```
>> det(T)
ans =
    72
```

Special cases:

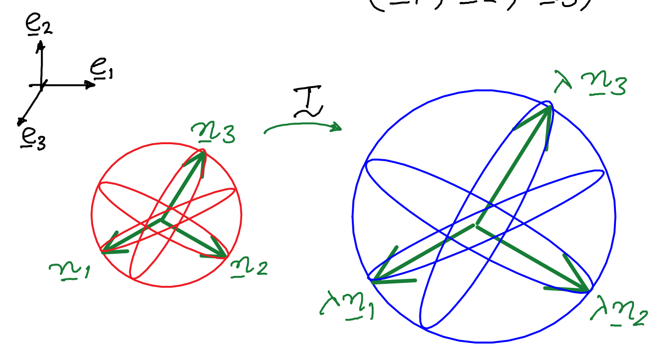
- Two roots repeated: $\lambda_1 = \lambda_2 \,,\; \lambda_3$

  $(\underset{\sim}{n}_1, \underset{\sim}{n}_2) \,,\; \underset{\sim}{n}_3$



- Three roots repeated: $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$

  $(\underset{\sim}{n}_1 \,,\, \underset{\sim}{n}_2 \,,\, \underset{\sim}{n}_3)$

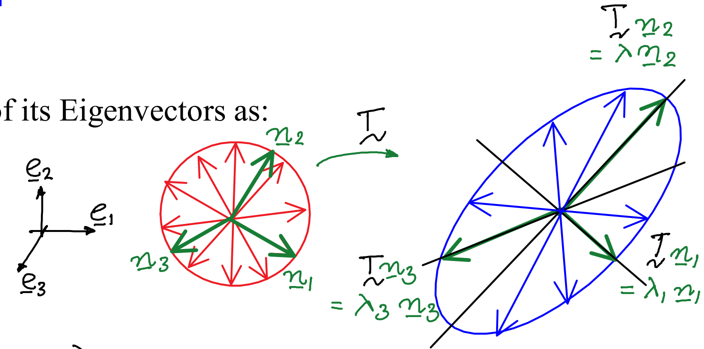


<u>Spectral decomposition of a (symmetric) tensor:</u>

A symmetric tensor ***T*** can be expressed in terms of its Eigenvectors as:

$$\underset{\sim}{T} = \sum_{i=1}^{3} \lambda_i \left(\underset{\sim}{n}_i \otimes \underset{\sim}{n}_i\right)$$

(Exception to summation convention)



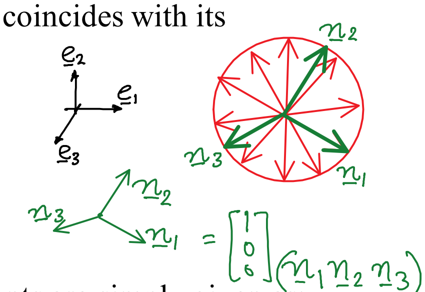
$$\text{i.e.} \quad \underset{\sim}{T} = \lambda_1 \left(\underset{\sim}{n}_1 \otimes \underset{\sim}{n}_1\right) + \lambda_2 \left(\underset{\sim}{n}_2 \otimes \underset{\sim}{n}_2\right) + \lambda_3 \, \underset{\sim}{n}_3 \otimes \underset{\sim}{n}_3$$

$$\text{i.e.} \quad \underset{\sim}{T} \sim \lambda_1 \begin{bmatrix} \\ n_1 \\ \\ \end{bmatrix} \begin{bmatrix} & n_1 & \end{bmatrix} + \lambda_2 \begin{bmatrix} \\ n_2 \\ \\ \end{bmatrix} \begin{bmatrix} & n_2 & \end{bmatrix} + \lambda_3 \begin{bmatrix} \\ n_3 \\ \\ \end{bmatrix} \begin{bmatrix} & n_3 & \end{bmatrix}$$

$$\text{Thus} \quad \underset{\sim}{T}\,\underset{\sim}{u} = \sum_{i=1}^{3} \lambda_i \left(\underset{\sim}{n}_i \otimes \underset{\sim}{n}_i\right) \underset{\sim}{u} = \sum_{i=1}^{3} \lambda_i \left(\underset{\sim}{n}_i \cdot \underset{\sim}{u}\right) \underset{\sim}{n}_i$$

<u>Note</u>: If we express the components of a tensor in a coordinate system that coincides with its principal eigenvectors: i.e. $\underset{\sim}{e}'_i = \underset{\sim}{n}_i$

Then using Spectral representation: $\underset{\sim}{T} \sim \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix}$



Also note that using this basis (called <u>canonical</u> basis), the principal invariants are simply given as:

$$I_T = \lambda_1 + \lambda_2 + \lambda_3$$

$$II_T = \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1$$

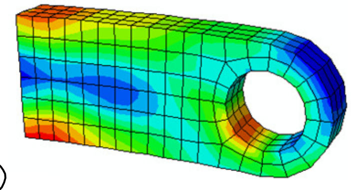
$$III_T = \lambda_1 \lambda_2 \lambda_3$$

<u>Caley-Hamilton Theorem</u>

An important property of tensors (and matrices) is that they satisfy their own characteristic equation:

$$-\underset{\sim}{T}^3 + I_T \, \underset{\sim}{T}^2 - II_T \, \underset{\sim}{T} + III_T \, \underset{\sim}{I} = \underset{\sim}{0}$$

Calculus of Scalars, Vectors and Tensors

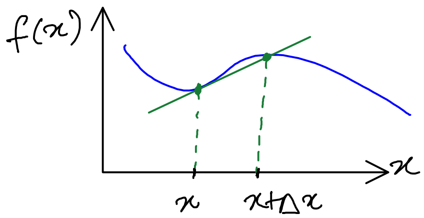


As previously noted, scalars, vectors and tensors are quantities that are associated with each point in a body as a field.
They can be expressed as functions of position:

$\theta(\underset{\sim}{x})$ (scalar)
$\underset{\sim}{v}(\underset{\sim}{x})$ (vector)
$\underset{\sim}{S}(\underset{\sim}{x})$ (tensor)

In order to work with fields, we need to use concepts of differential and integral calculus.
First recall the following basic concepts in 1D

Scalar function of 1 variable:



$n_- \cong -1$ $\quad$ $n_+ \cong +1$

Derivative:

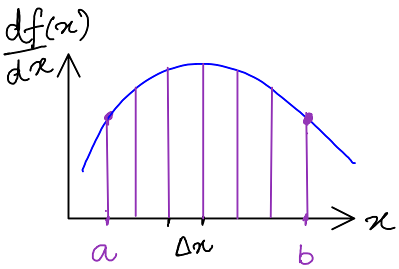
$$\frac{df}{dx} \equiv \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Note:

FLUX of $f(x)$ over the boundary of Δx

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{\boxed{f(x+\Delta x)\, n_+ + f(x)\, n_-}}{\boxed{\Delta x}}$$

measure (length) of Δx

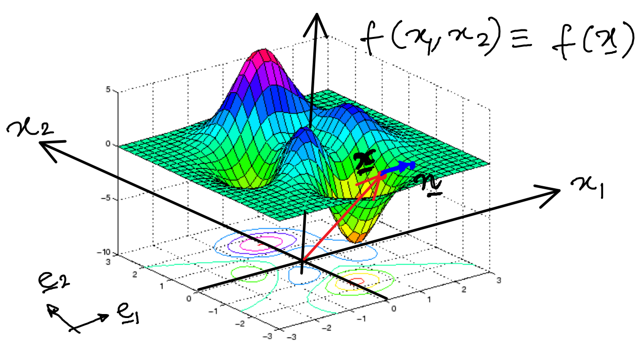


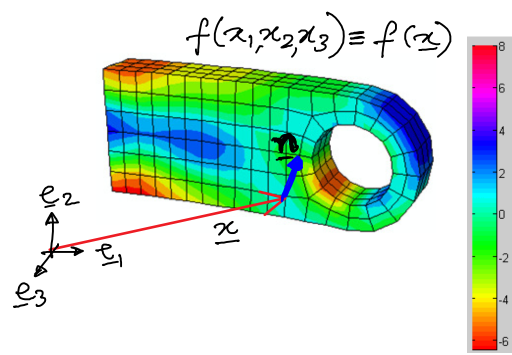
Fundamental Theorem of Calculus:

$$\int_a^b \underbrace{\left(\frac{df}{dx}\right)}_{f'(x)} dx \equiv \lim_{N\to\infty} \sum_{i=0}^{N} f'(a + i\Delta x)\Delta x = f(b) - f(a)$$

where $\Delta x = \frac{b-a}{N}$ $\quad = f(b)\, n_+ + f(a)\, n_-$

Scalar field: scalar function of position "vector" in 2D / 3D





Gateaux

Directional Derivative

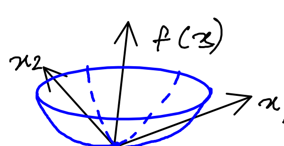
(of a scalar field,
at a specific point $\boldsymbol{x}$,
in the direction $\boldsymbol{n}$)

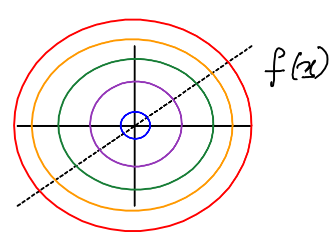
$$D f(\underset{\sim}{x}) \cdot \underset{\sim}{n} \equiv \lim_{\epsilon \to 0} \frac{f(\underset{\sim}{x} + \epsilon \underset{\sim}{n}) - f(\underset{\sim}{x})}{\epsilon} \qquad (= \text{scalar})$$

$$\boxed{Df(x) \cdot \underset{\sim}{n} = \left[\frac{d}{d\epsilon} f(\underset{\sim}{x} + \epsilon \underset{\sim}{n})\right]\Big|_{\epsilon=0}}$$

Example: $f(\underset{\sim}{x}) = \underset{\sim}{x} \cdot \underset{\sim}{x}$

$x_1 x_1 + x_2 x_2 + x_3 x_3$





$$D f(\underset{\sim}{x}) \cdot \underset{\sim}{n} = \frac{d}{d\epsilon}\left[(\underset{\sim}{x} + \epsilon \underset{\sim}{n}) \cdot (\underset{\sim}{x} + \epsilon \underset{\sim}{n})\right]\Big|_{\epsilon=0}$$

$$= \left[\frac{d}{d\epsilon}\left[\underbrace{\cancel{\underset{\sim}{x} \cdot \underset{\sim}{x}}}_{0} + 2\underbrace{\cancel{\epsilon}}_{1}\, \underset{\sim}{x} \cdot \underset{\sim}{n} + \underbrace{\cancel{\epsilon^2}}_{2\epsilon_{\epsilon=0}}\, \underset{\sim}{n} \cdot \underset{\sim}{n}\right]\right]\Big|_{\epsilon=0} = 2\, \underset{\sim}{x} \cdot \underset{\sim}{n}$$

Example

$\sim 2\,(1,1) \cdot (1,1) = 4$

Gradient of a scalar field: Direction of maximum change (increase) in value of $f(\boldsymbol{x})$:
(results in a vector field)

$$\underline{\nabla} f(\underline{x}) \equiv \frac{\partial f(\underline{x})}{\partial x_i}\underline{e}_i$$

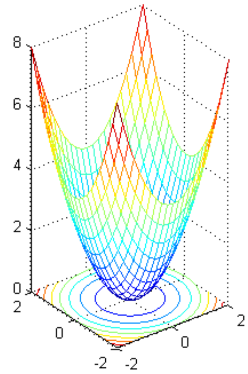
Example: $f(\underline{x}) = \underline{x}\cdot\underline{x} = x_i x_i$

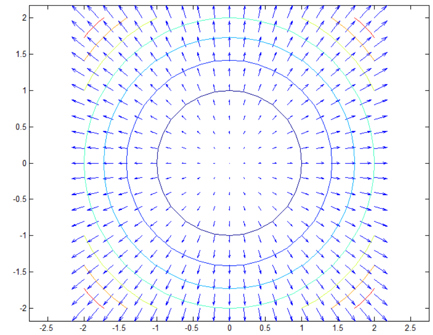
$$\underline{\nabla} f(x) = \frac{\partial f}{\partial x_j}\underline{e}_j = \frac{\partial(x_i x_i)}{\partial x_j}\underline{e}_j$$

$$= \Big(\underbrace{\frac{\partial x_i}{\partial x_j}}_{\delta_{ij}} x_i + x_i \underbrace{\frac{\partial x_i}{\partial x_j}}_{\delta_{ij}}\Big)\underline{e}_j$$

$$= 2\, x_i\, \cancel{\delta_{ij}}\, \underline{e}_{\cancel{j}\,i} = 2 x_i \underline{e}_i$$

$$= 2\underline{x}$$

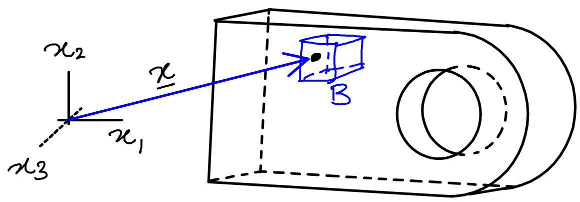




Coordinate independent representation of the gradient of a scalar field:

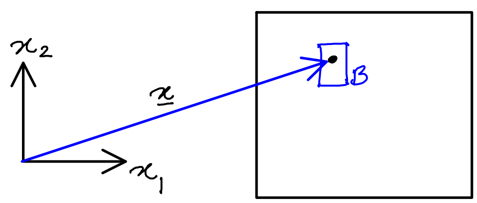
In 3D:

$$\underline{\nabla} f(\underline{x}) \equiv \lim_{Vol(B)\to 0}\frac{1}{Vol(B)}\int_{Area(B)} f\,\underline{n}\,da$$



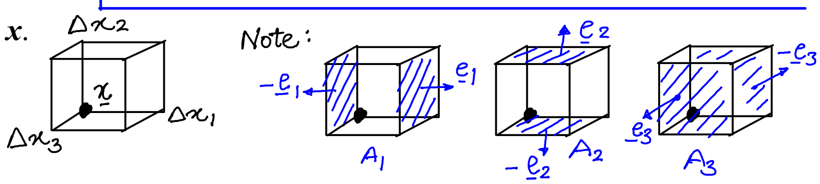
In 2D:

$$\underline{\nabla} f(\underline{x}) \equiv \lim_{Area(B)\to 0}\frac{1}{Area(B)}\int_{Boundary(B)} f\,\underline{n}\,ds$$



B can be any infinitesimal region enclosing $\boldsymbol{x}$.

Choose B:



$Vol(B) = \Delta x_1 \Delta x_2 \Delta x_3$

$Area(B) = 2(\Delta x_1 \Delta x_2 + \Delta x_2 \Delta x_3 + \Delta x_1 \Delta x_3)$

Note: $$\int_{Area(B)} f\,\underline{n}\,da = \sum_{i=1}^{3}\left[\int_{A_i} f(\underline{x}+\Delta x_i \underline{e}_i)\,\underline{e}_i\,dA + \int_{A_i} f(\underline{x})(-\underline{e}_i)\,dA\right]$$

$$= \sum_{i=1}^{3}\left[\bar{f}(\underline{x}+\Delta x_i \underline{e}_i) - \bar{f}(\underline{x})\right]\underline{e}_i\, A_i \qquad \text{where } \bar{f} \equiv \frac{1}{A_i}\int f\, dA$$

Note $Vol(B) = A_1 \Delta x_1 = A_2 \Delta x_2 = A_3 x_3 = A_i \Delta x_i$ (no sum)

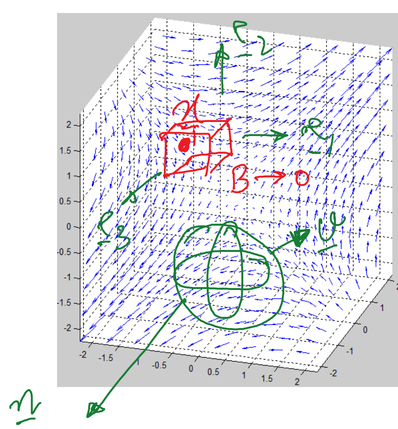
Thus $$\underline{\nabla} f = \sum_{i=1}^{3}\left[\lim_{\Delta x_i \to 0}\frac{\left[\bar{f}(\underline{x}+\Delta x_i \underline{e}_i) - \bar{f}(\underline{x})\right]\underline{e}_i\,\cancel{A_i}}{\cancel{A_i}\,\Delta x_i}\right] = \boxed{\sum_{i=1}^{3}\frac{\partial f}{\partial x_i}\underline{e}_i}$$

Vector field: vector function of the position vector $\boldsymbol{x}$ in 2D/3D:
Example

$$\underline{v}(\underline{x}) = (x_2+x_3)\underline{e}_1 + (x_1+x_3)\underline{e}_2 + (x_1+x_2)\underline{e}_3$$

i.e. $$\underline{v}(\underline{x}) \sim \begin{bmatrix} x_2+x_3 \\ x_1+x_3 \\ x_1+x_2 \end{bmatrix}$$

$(\underline{v}\cdot\underline{n})$



Divergence of a vector field: (results in scalar field)

Measures the change in 'flux' (outflow-inflow) at each point in a vector field.

$$\operatorname{div} \underline{v}(\underline{x}) \equiv \lim_{Vol(B)\to 0} \frac{1}{Vol(B)} \int_{Area(B)} \underline{v}(\underline{x}) \cdot \underline{n}\, dA \qquad \text{Note: } \nabla \sim \begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{bmatrix}$$

$$\operatorname{div} \underline{v}(\underline{x}) = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = \frac{\partial v_i(\underline{x})}{\partial x_i} = v_{i,i} = \boxed{\nabla \cdot \underline{v}(\underline{x})} \text{ or } (\underline{v}(\underline{x}) \cdot \nabla)$$

Comma $\Rightarrow \frac{\partial}{\partial x}$

Examples:

$\rightarrow$ $\underline{v}(\underline{x}) = (x_2+x_3)\underline{e}_1 + (x_1+x_3)\underline{e}_2 + (x_1+x_2)\underline{e}_3 \Rightarrow \operatorname{div} \underline{v}(\underline{x}) = 0$

$\rightarrow$ $\underline{v}(\underline{x}) = x_1\underline{e}_1 + x_2\underline{e}_2 + x_3\underline{e}_3 = \underline{x} \Rightarrow \operatorname{div} \underline{v}(\underline{x}) = 1+1+1 = 3$

$\rightarrow$ $\underline{v}(\underline{x}) = x_1^2\underline{e}_1 + x_2^2\underline{e}_2 + x_3^2\underline{e}_3 \Rightarrow \operatorname{div} \underline{v}(\underline{x}) = 2(x_1+x_2+x_3)$

Curl of a vector field: (results in a vector field)

Measures change in 'circulation' at a point in a vector field.

$$\operatorname{curl} \underline{v}(\underline{x}) = \lim_{Vol(B)\to 0} \frac{1}{Vol(B)} \int_{Area(B)} \underline{v} \times \underline{n}\, da \qquad \frac{\partial v_i}{\partial x_j}$$

$$\operatorname{curl} \underline{v}(\underline{x}) = \boxed{-\nabla \times \underline{v}(\underline{x})} = -\left(\frac{\partial}{\partial x_j}\underline{e}_j\right) \times v_i(\underline{x})\, \underline{e}_i = \epsilon_{ijk}\, v_{i,j}\, \underline{e}_k$$

(or $\underline{v}(\underline{x}) \times \nabla$)

Examples:

$\rightarrow$ $\underline{v}(\underline{x}) = \underline{x} \Rightarrow -\nabla \times \underline{v}(\underline{x}) = \epsilon_{ijk} \frac{\partial x_i}{\partial x_j} \underline{e}_k = \epsilon_{\cancel{i}jk}\, \delta_{i\cancel{j}}\, \underline{e}_k = \underline{0}$

$\rightarrow$ $\underline{v}(\underline{x}) = (x_2-x_3)\underline{e}_1 + (x_3-x_1)\underline{e}_2 + (x_1-x_2)\underline{e}_3$

$$\Rightarrow -(\nabla \times \underline{v}(\underline{x})) = \begin{bmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ (x_2-x_3) & (x_3-x_1) & (x_1-x_2) \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \end{bmatrix} = \left(\frac{\partial}{\partial x_3}(\overset{1}{\cancel{x_3}}-x_1) - \frac{\partial}{\partial x_2}(x_1-\overset{-1}{\cancel{x_2}})\right)\underline{e}_1$$

(or $\underline{v}(\underline{x}) \times \nabla$)

$$+\left(\frac{\partial}{\partial x_1}(\overset{1}{\cancel{x_1}}-x_2) - \frac{\partial}{\partial x_3}(x_2-\overset{-1}{\cancel{x_3}})\right)\underline{e}_2$$

$$= 2(\underline{e}_1+\underline{e}_2+\underline{e}_3) \qquad +\left(\frac{\partial}{\partial x_2}(\overset{1}{\cancel{x_2}}-x_3) - \frac{\partial}{\partial x_1}(x_3-\overset{-1}{\cancel{x_1}})\right)\underline{e}_3$$

Gradient of a vector field: (results in a tensor field)

Measures rate of change of vector field in all possible directions.

$$\nabla \underline{v}(\underline{x}) \equiv \lim_{Vol(B)\to 0} \frac{1}{Vol(B)} \int_{Area(B)} \underline{v} \otimes \underline{n}\, dA \qquad \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \end{bmatrix}$$

(or $\underline{v}(\underline{x}) \otimes \nabla$)

$$\nabla \underline{v}(\underline{x}) = \boxed{\frac{\partial v_i}{\partial x_j}\, \underline{e}_i \otimes \underline{e}_j} \sim \begin{bmatrix} v_{1,1} & v_{1,2} & v_{1,3} \\ v_{2,1} & v_{2,2} & v_{2,3} \\ v_{3,1} & v_{3,2} & v_{3,3} \end{bmatrix}$$

Examples:

$\rightarrow$ $\underline{v}(\underline{x}) = \underline{x} \Rightarrow \nabla \underline{v}(\underline{x}) = \frac{\partial x_i}{\partial x_j}\, \underline{e}_i \otimes \underline{e}_j \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

δ_{ij}

$\rightarrow$ Look at Example 7 in the book!

Analogies between vector products and vector field derivatives:

$$\begin{array}{llll} \rightarrow \text{Dot}: & \underline{u}\cdot\underline{v} \; ; & \underline{v}\cdot\underline{\nabla} & (\text{div}) \\ \rightarrow \text{Cross}: & \underline{u}\times\underline{v} \; ; & \underline{v}\times\underline{\nabla} & (\text{curl}) \\ \rightarrow \text{Dyad}: & \underline{u}\otimes\underline{v} \; ; & \underline{v}\otimes\underline{\nabla} & (\text{grad}) \end{array}$$

<u>Directional derivative of a vector field</u> (gives a vector)

Rate of change of a vector field at a point ***x*** in a specific direction ***n***

$$D\underline{v}\cdot\underline{n} = \lim_{\epsilon\to 0}\frac{\underline{v}(\underline{x}+\epsilon\underline{n}) - \underline{v}(\underline{x})}{\epsilon} = \left[\frac{d}{d\epsilon}\underline{v}(\underline{x}+\epsilon\underline{n})\right]\Bigg|_{\epsilon=0} = \nabla\underset{\sim}{v}\,\underline{n}$$

Examples:

For $\underline{v}(\underline{x}) = \underline{x}$ $\rightarrow$ $D\underline{v}(\underline{x})\cdot\underline{x} = \underset{\sim}{I}\,\underline{x} = \underline{x}$

$\rightarrow$ $D\underline{v}(\underline{x})\cdot\underline{x}_1 = \underset{\sim}{I}\,\underline{x}_1 = \underline{x}_1$

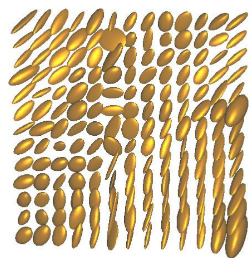
(where $\underline{x}_1$ is a vector $\perp$ to $\underline{x}$)

<u>Tensor field: tensor function of the position "vector"</u>

Example:

$$\underset{\sim}{T}(\underline{x}) = (\underline{x}\cdot\underline{x})\underset{\sim}{I} - 2\,\underline{x}\otimes\underline{x}$$

Note: A tensor ***T***(***x***) may be a non-linear function ***x***, but its action on a vector ***u***(***x***) is still linear: $\underset{\sim}{T}\,\underline{u}$



http://www.mathworks.com/

<u>Divergence of a tensor field</u>: (results in a vector field)

$$\text{div}(\underset{\sim}{T}) \equiv \lim_{Vol(B)\to 0}\frac{1}{Vol(B)}\int_{Area(B)}\underset{\sim}{T}\,\underline{n}\,dA \qquad (=\underset{\sim}{T}\,\underline{\nabla}) \qquad T_{ij,j}$$

$$\text{div}(\underset{\sim}{T}) = \frac{\partial\underset{\sim}{T}}{\partial x_i}\underline{e}_i = \frac{\partial}{\partial x_i}\left(T_{j\cancel{k}_{\,i}}\,\underline{e}_j\otimes\underline{e}_k\right)\underline{e}_i = \frac{\partial T_{ji}}{\partial x_i}\underline{e}_j = \boxed{\frac{\partial T_{ij}}{\partial x_j}\underline{e}_i}$$

with $\underbrace{\underline{e}_k)\,\underline{e}_i}_{\delta_{\cancel{k}i}}$ (replace $i \leftrightarrow j$)

Example:

$$\underset{\sim}{T} = (\underline{x}\cdot\underline{x})\underset{\sim}{I} - 2\,\underline{x}\otimes\underline{x}$$

$$= x_k x_k\,\delta_{ij}\,\underline{e}_i\otimes\underline{e}_j - 2\,x_i x_j\,\underline{e}_i\otimes\underline{e}_j$$

(Note: The 2 terms follow indicial notation individually)

$$\text{div}(\underset{\sim}{T}) = \frac{\partial(x_k x_k)}{\partial x_{\cancel{j}\,i}}\delta_{i\cancel{j}}\,\underline{e}_i - 2\,\frac{\partial(x_i x_j)}{\partial x_j}\underline{e}_i$$

$$= 2\,x_{\cancel{k}\,i}\,\delta_{\cancel{k}i}\,\underline{e}_i - 2\times\left(\delta_{i\cancel{j}}\,x_{\cancel{j}\,i} + x_i\,\cancel{\delta_{jj}}^{\,3}\right)\underline{e}_i$$

$$= 2\,x_i\,\underline{e}_i - 2\,(4x_i)\,\underline{e}_i$$

$$= -6\,x_i\,\underline{e}_i = -6\,\underline{x}$$

Integral Theorems

These theorems are generalized versions of the fundamental theorem of calculus.

Divergence theorem (Gauss theorem)

- For vector fields. $$\int_{Vol(B)} \operatorname{div} \underline{v}(\underline{x})\, dV = \int_{Area(B)} \underline{v}(\underline{x})\cdot \underline{n}\, da$$

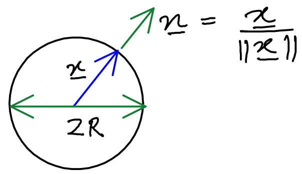
unit normal to Boundary; Vol(B) ← Region; Area(B) ← Boundary

(for any region / subregion B)

Example:

Consider $\underline{v}(\underline{x}) = \underline{x}$; $\operatorname{div} \underline{v}(\underline{x}) = x_{i,i} = 3$

Over a sphere:



$$\int_{Vol} 3\, dV = \int_{Area} \underline{x}\cdot \boxed{\frac{\underline{x}}{\|\underline{x}\|}}^{\,n} da$$

$$\cancel{3}\left(\frac{4}{\cancel{3}}\pi R^3\right) = \int_{Area} \frac{R^2}{R}\, da = R\left(4\pi R^2\right)$$

(Verified!)

- For gradient of a scalar field: $$\int_{Vol(B)} \nabla g\, dV = \int_{Area(B)} g\, \underline{n}\, da$$

- For gradient of a vector field: $$\int_{Vol(B)} \nabla \underline{v}\, dV = \int_{Area(B)} \underline{v} \otimes \underline{n}\, da$$

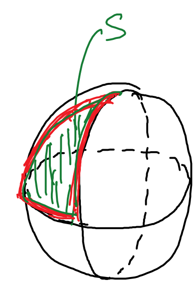
- For a tensor field: $$\int_{Vol(B)} \operatorname{div}(\underset{\sim}{T})\, dV = \int_{Area(B)} \underset{\sim}{T}\, \underline{n}\, da$$

Curl Theorem (special case of the Stokes theorem):

c.f. http://mathworld.wolfram.com/CurlTheorem.html

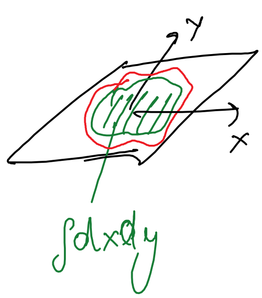
$$\int_{Surface(S)} (\nabla \times \underline{v})\, ds = \oint_{Boundary(S)} \underline{v}\cdot d\underline{x}$$

(line integral)



Green's Theorem

(special case of Curl theorem / Stokes' theorem for a plane)



$$\int_{Area(B)} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = \oint_{Boundary(B)} M\, dx + N\, dy$$