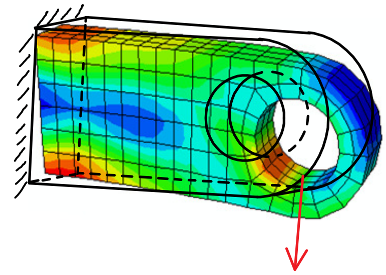
Chapter 2: Kinematics of Deformation

In this chapter, we will study how bodies/structures move/deform and how can this motion/deformation be described mathematically.
(In general, bodies/structures move/deform when forces are acting on them, but we are not concerned (for now) about the causes of this motion/deformation.)
We are concerned only about describing the motion/deformation.

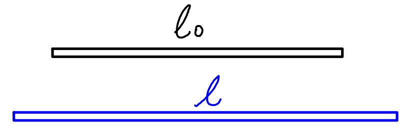


Motion / deformation can be:

- Fast: Dynamic effects are important: Acceleration, inertia etc.
- Slow: Dynamic effects can be neglected: (quasi-)static. Or after *steady state* has been achieved.

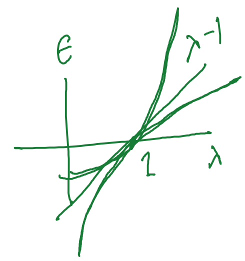
Stretch of a material in 1D

Consider a uniform bar of some material before and after motion/deformation.



Stretch (λ) $\boxed{l = \lambda\, l_0}$ i.e. $\lambda = \frac{l}{l_0}$

Conventional notions of strain in 1D



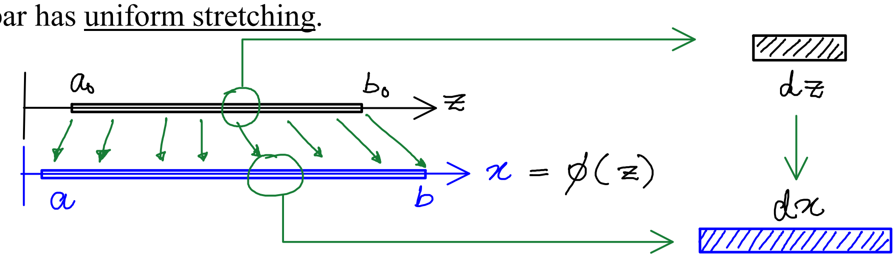
- Engineering strain $\quad \epsilon_{ENG} \equiv \frac{\Delta l}{l_0} = \frac{l - l_0}{l_0} = \lambda - 1$
- Natural (true) strain $\quad \epsilon_{NAT/TRUE} \equiv \frac{\Delta l}{l} = \frac{l - l_0}{l} = 1 - \frac{1}{\lambda}$
- Green-Lagrangian strain $\quad E \equiv \frac{1}{2}\left(\frac{l^2 - l_0^2}{l_0^2}\right) = \frac{1}{2}(\lambda^2 - 1) = \frac{1}{2}(\lambda-1)(\lambda+1)$
- Almansi-Eulerian strain $\quad e = \frac{1}{2}\left(\frac{l^2 - l_0^2}{l^2}\right) = \frac{1}{2}\left(1 - \frac{1}{\lambda^2}\right) = \frac{1}{2}\left(1 - \frac{1}{\lambda}\right)\left(1 + \frac{1}{\lambda}\right)$
- Logarithmic strain $\quad \epsilon_{ln} = \ln(\lambda)$

Note: $\sum_{i=1}^{\infty} \frac{\Delta l_i}{l_i} = \int_{l_0}^{l} \frac{1}{l}\, dl = \ln(l) - \ln(l_0) = \ln\left(\frac{l}{l_0}\right) = \ln(\lambda)$

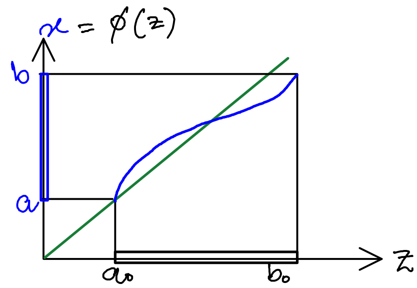
All these are average measures of strain (for the entire bar) that are applicable for cases when the bar has uniform stretching.

General definition of strains in 1D:

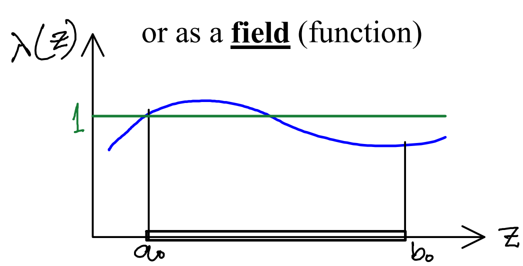
(For non-uniform stretch)



Can be represented as a **map**:



or as a **field** (function)



$\boxed{dx = \lambda(z)\, dz}$

$\boxed{\lambda(z) = \frac{dx}{dz}}$

Deformation maps $\phi(z)$ and displacement vector fields $u(z)$ in 3D

Generalizes the 1D concept of the map to 3D.
Takes the position vector z of any point in the undeformed configuration and Return its position in the deformed configuration.

$$x = \phi(z)$$

$$u(z) = x - z$$

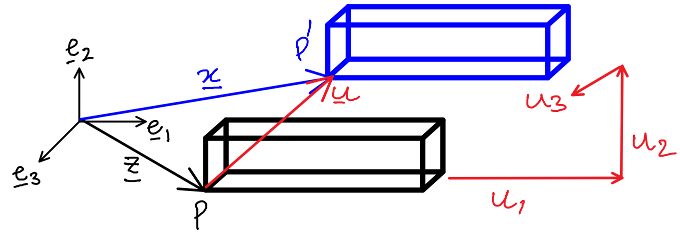
Deformation map $x = \phi(z)$

$x = z + u(z)$ — Displacement field

Examples of deformation maps:

(i) Translation $x = z + u$

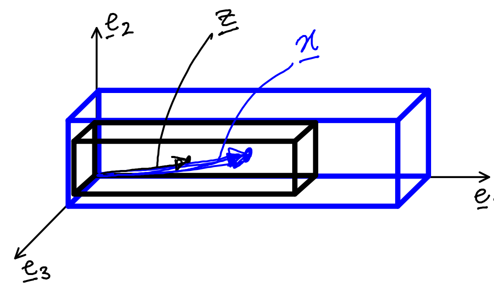
$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} z_1 \\ z_2 \\ z_3 \end{Bmatrix} + \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$



(ii) Uniform Expansion in all 3 directions

$$x = \phi(z) = \alpha z$$

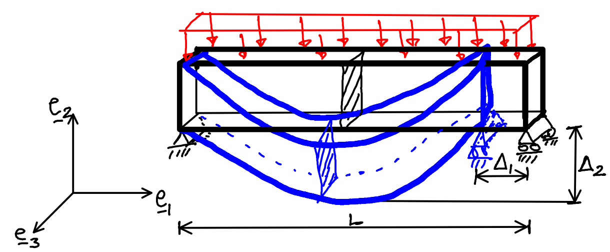
$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \alpha \begin{Bmatrix} z_1 \\ z_2 \\ z_3 \end{Bmatrix}$$



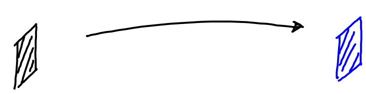
(iii) Approximate bending deformation

$$x = \phi(z)$$

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} \phi_1(z_1, z_2, z_3) \\ \phi_2(z_1, z_2, z_3) \\ \phi_3(z_1, z_2, z_3) \end{Bmatrix}$$



$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} z_1 - \frac{z_1}{L}\Delta_1 \\ z_2 - \Delta_2 \sin\left(\frac{z_1}{L}\pi\right) \\ z_3 \end{Bmatrix}$$



Verify: for a point on the mid cross-section: $\left(\frac{L}{2}, z_2, z_3\right) \rightarrow \left(\frac{L}{2} - \frac{\Delta_1}{2};\ z_2 - \Delta_2;\ z_3\right)$

(iv) Pure bending of a prismatic cantilever beam:
(pages 250-255, Timoshenko & Goodier)

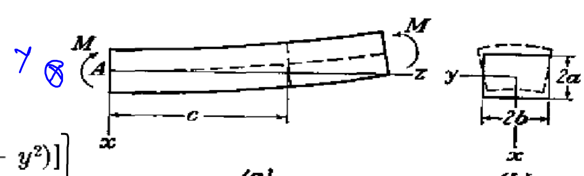


FIG 141.

$$x = z + u(z)$$

$$x \sim \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} \quad z \sim \begin{Bmatrix} z_1 \to x \\ z_2 \to y \\ z_3 \to z \end{Bmatrix} \quad u \sim \begin{Bmatrix} u = -\frac{1}{2R}[z^2 + \nu(x^2 - y^2)] \\ v = -\frac{\nu x y}{R} \\ w = \frac{xz}{R} \end{Bmatrix}$$

Note: For a cross section at $z = c$:

$$x_3 = c + w = c + \frac{cx}{R}$$

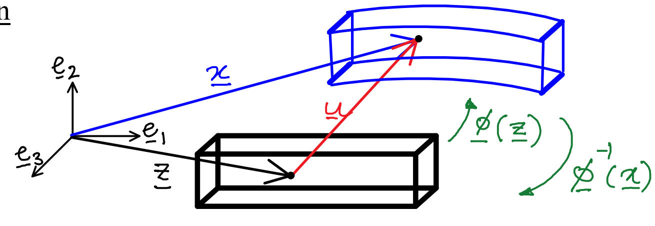
Note: For the lateral surfaces of the beam:

$$x_2 = \pm b + v = \pm b\left(1 - \frac{\nu x}{R}\right)$$

$$x_1 = \pm a + u = \pm a - \frac{1}{2R}[c^2 + \nu(a^2 - y^2)]$$

Lagrangian vs. Eulerian descriptions of motion/deformation

Note: The displacement field can be expressed as:

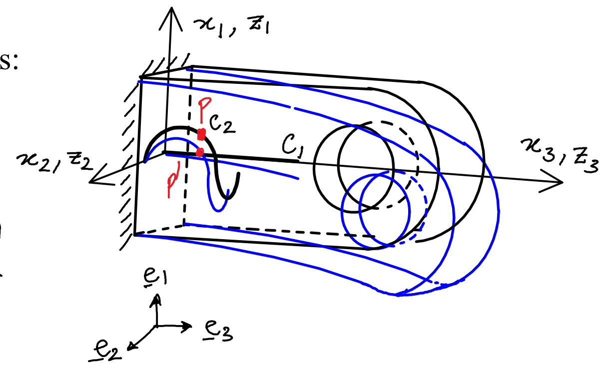


$$\boldsymbol{u} = \boldsymbol{u}_L(\boldsymbol{z}) = \boldsymbol{x}(\boldsymbol{z}) - \boldsymbol{z} = \boldsymbol{\phi}(\boldsymbol{z}) - \boldsymbol{z} \quad \text{(Lagrangian)}$$

$$\boldsymbol{u} = \boldsymbol{u}_E(\boldsymbol{x}) = \boldsymbol{x} - \boldsymbol{z}(\boldsymbol{x}) = \boldsymbol{x} - \boldsymbol{\phi}^{-1}(\boldsymbol{x}) \quad \text{(Eulerian)}$$

Stretch along a curve in 3D

To generalize the ideas of stretch and strain to 3D consider a curve C embedded in a structure as it deforms:



Examples: (curve parameter s)

$$\underline{z}(s) \sim \begin{Bmatrix} z_1(s) \\ z_2(s) \\ z_3(s) \end{Bmatrix}$$

$$C_1 = \left\{ \underline{z} : \underline{z} \sim \begin{Bmatrix} 0 \\ 0 \\ s \end{Bmatrix} ;\ s \in [0, 1] \right\}$$

$$C_2 = \left\{ \underline{z} : \underline{z} \sim \begin{Bmatrix} R \sin(s/L \; 2\pi) \\ R\cos(s/L \; 2\pi) \\ s \end{Bmatrix} ;\ s \in [0, L] \right\}$$

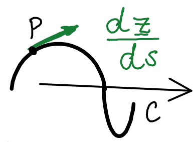
...infinitely many possible curves.

The deformed locations of these curves are given by:

$$\phi(C_1) = \{ \underline{x} : \underline{x} = \underline{\phi}(\underline{z}) ;\ \underline{z} \in C_1 \}$$

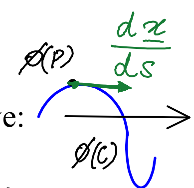
$$\phi(C_2) = \{ \underline{x} : \underline{x} = \underline{\phi}(\underline{z}) ;\ \underline{z} \in C_2 \}$$

To find the stretch along the curve:
Tangent to the undeformed curve:



$$\frac{d\underline{z}}{ds} = \lim_{\Delta s \to 0} \frac{\underline{z}(s+\Delta s) - \underline{z}(s)}{\Delta s}$$

Tangent to the ~~un~~deformed curve:



$$\frac{d\underline{x}}{ds} = \lim_{\Delta s \to 0} \frac{\underline{x}(s+\Delta s) - \underline{x}(s)}{\Delta s}$$

The 1-D stretch at a point P along the curve C is given by:

$$\boxed{\left\| \frac{d\underline{x}}{ds} \right\| = \lambda(s) \left\| \frac{d\underline{z}}{ds} \right\|}$$

Note that the stretch at P in an arbitrary direction can be obtained by using a different curve passing though P.

Also note:

$$\frac{d\underline{x}}{ds} \sim \begin{Bmatrix} \frac{dx_1}{ds} \\ \frac{dx_2}{ds} \\ \frac{dx_3}{ds} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial x_1}{\partial z_1} \cdot \frac{dz_1}{ds} + \frac{\partial x_1}{\partial z_2} \cdot \frac{dz_2}{ds} + \frac{\partial x_1}{\partial z_3} \cdot \frac{dz_3}{ds} \\ \frac{\partial x_2}{\partial z_1} \cdot \frac{dz_1}{ds} + \frac{\partial x_2}{\partial z_2} \cdot \frac{dz_2}{ds} + \frac{\partial x_1}{\partial z_3} \cdot \frac{dz_3}{ds} \\ \frac{\partial x_3}{\partial z_1} \cdot \frac{dz_1}{ds} + \frac{\partial x_3}{\partial z_2} \cdot \frac{dz_2}{ds} + \frac{\partial x_3}{\partial z_3} \cdot \frac{dz_3}{ds} \end{Bmatrix}$$

$$\frac{d\underline{x}}{ds} \sim \begin{bmatrix} \frac{\partial x_1}{\partial z_1} & \frac{\partial x_1}{\partial z_2} & \frac{\partial x_1}{\partial z_3} \\ \frac{\partial x_2}{\partial z_1} & \frac{\partial x_2}{\partial z_2} & \frac{\partial x_2}{\partial z_3} \\ \frac{\partial x_3}{\partial z_1} & \frac{\partial x_3}{\partial z_2} & \frac{\partial x_3}{\partial z_3} \end{bmatrix} \begin{bmatrix} \frac{dz_1}{ds} \\ \frac{dz_2}{ds} \\ \frac{dz_3}{ds} \end{bmatrix} \sim [\underline{\nabla}_z \underline{x}] \frac{d\underline{z}}{ds}$$

$$[\underline{x} \otimes \underline{\nabla}_z] \frac{d\underline{z}}{ds}$$

Deformation gradient tensor ***F***

The relationship for stretches in arbitrary directions in 3D can be expressed more compactly as:

$$\frac{d\underline{x}}{ds} = \frac{dx_i}{ds}\underline{e}_i = \frac{\partial x_i}{\partial z_j}\cdot\frac{dz_j}{ds}\underline{e}_i = \left[\frac{\partial x_i}{\partial z_j}(\underline{e}_i \otimes \underline{g}_j)\right]\left(\frac{dz_k}{ds}\underline{g}_k\right)$$

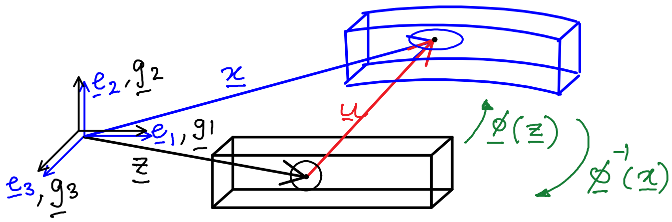
$$\Rightarrow \boxed{\frac{d\underline{x}}{ds} = \underset{\sim}{F}\,\frac{d\underline{z}}{ds}} \quad \text{where } \underset{\sim}{F} \equiv \underset{\sim}{\nabla}_{\underline{z}}\,\underline{x} \equiv (\underline{x}\otimes\underline{\nabla}_{\underline{z}}) \equiv \frac{\partial\underline{x}}{\partial\underline{z}}$$

$$\text{Since } \underline{x} = \underline{\phi}(\underline{z}) \Rightarrow \underset{\sim}{F}(\underline{z}) = \underset{\sim}{\nabla}_{\underline{z}}\,\underline{\phi}(\underline{z}) = (\underline{\phi}(\underline{z})\otimes\underline{\nabla}_{\underline{z}}) = \frac{\partial\underline{\phi}(\underline{z})}{\partial\underline{z}}$$

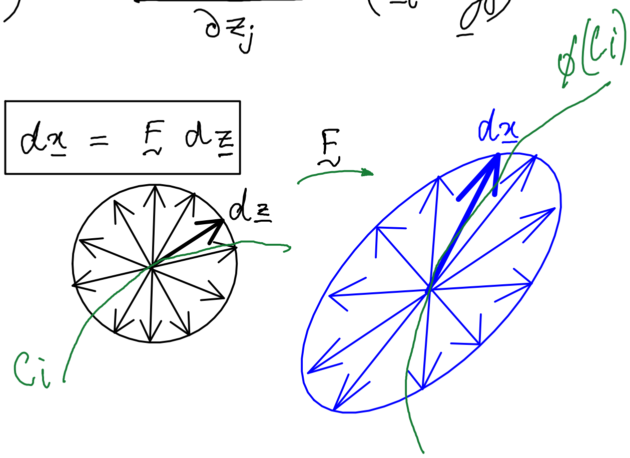
In components:

$$\underset{\sim}{F}(\underline{z}) = \phi_{i,j}(\underline{e}_i\otimes\underline{g}_j) = \frac{\partial\phi_i(z_1,z_2,z_3)}{\partial z_j}(\underline{e}_i\otimes\underline{g}_j)$$

A useful interpretation of ***F***



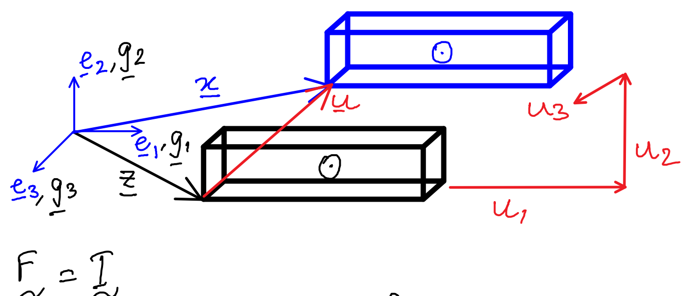
$$\boxed{d\underline{x} = \underset{\sim}{F}\,d\underline{z}}$$



Examples:

(i) Translation $\underline{x} = \underline{\phi}(\underline{z}) = \underline{z} + \underline{u}$

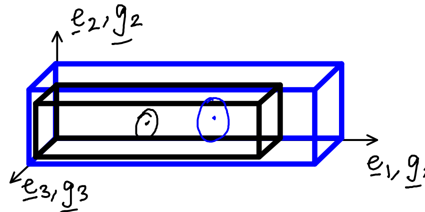
$$\begin{Bmatrix}x_1\\x_2\\x_3\end{Bmatrix} = \begin{Bmatrix}z_1+u_1\\z_2+u_2\\z_3+u_3\end{Bmatrix}, \quad \underset{\sim}{F}(\underline{z}) \sim \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$



$$\underset{\sim}{F} = \underset{\sim}{I}$$

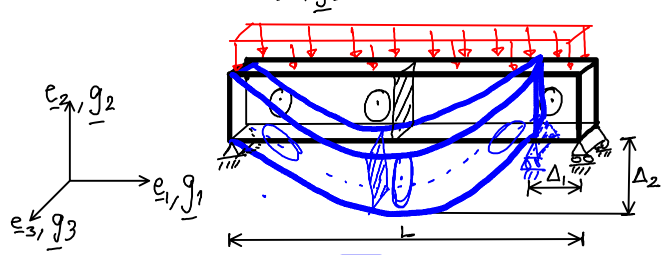
(ii) Uniform Expansion in all 3 directions

$$\begin{Bmatrix}x_1\\x_2\\x_3\end{Bmatrix} = \alpha\begin{Bmatrix}z_1\\z_2\\z_3\end{Bmatrix} \quad \underline{x} = \underline{\phi}(\underline{z}) = \alpha\underline{z} \Rightarrow \underset{\sim}{F} = \alpha\underset{\sim}{I}$$



(iii) Approximate bending deformation $\underline{x} = \underline{\phi}(\underline{z})$

$$\begin{Bmatrix}x_1\\x_2\\x_3\end{Bmatrix} = \begin{Bmatrix}z_1 - \frac{z_1}{L}\Delta_1\\ z_2 - \Delta_2\sin\left(\frac{z_1}{L}\pi\right)\\ z_3\end{Bmatrix}$$

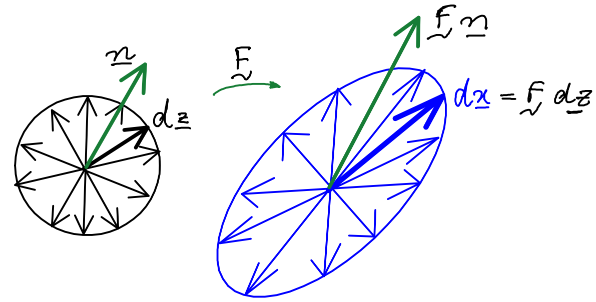


$$\Rightarrow \underset{\sim}{F}(\underline{z}) \sim \begin{bmatrix}1-\frac{\Delta_1}{L} & 0 & 0\\ -\Delta_2\cos\left(\frac{z_1}{L}\pi\right)\cdot\frac{\pi}{L} & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}1\\0\\0\end{bmatrix};\ \begin{bmatrix}0\\1\\0\end{bmatrix};\ \begin{bmatrix}0\\0\\1\end{bmatrix} \quad \begin{bmatrix}1-\frac{\Delta_1}{L}\\-\Delta_2\cos\ldots\\0\end{bmatrix}$$

Stretch and Strain in arbitrary directions in 3D

Using the interpretation of $\boldsymbol{F}$ as: $d\underset{\sim}{x} = \underset{\sim}{F}\, d\underset{\sim}{z}$
we can calculate the stretch in any arbitrary direction $\boldsymbol{n}$ of the undeformed configuration.



Stretch

$$\lambda(\underset{\sim}{n}) = \frac{\|\underset{\sim}{F}\underset{\sim}{n}\|}{\|\underset{\sim}{n}\| \to 1} = \sqrt{(\underset{\sim}{F}\underset{\sim}{n})\cdot(\underset{\sim}{F}\underset{\sim}{n})}$$

Alternatively,
$$\lambda^2(\underset{\sim}{n}) = \underset{\sim}{n}\cdot\underset{\sim}{F}^T\underset{\sim}{F}\underset{\sim}{n}$$
$$\boxed{\lambda^2(\underset{\sim}{n}) = \underset{\sim}{n}\cdot\underset{\sim}{C}\underset{\sim}{n}} \qquad \text{where} \qquad \boxed{\underset{\sim}{C} \equiv \underset{\sim}{F}^T\underset{\sim}{F}}$$

$\underset{\sim}{C}$: Right Cauchy-Green Deformation Tensor

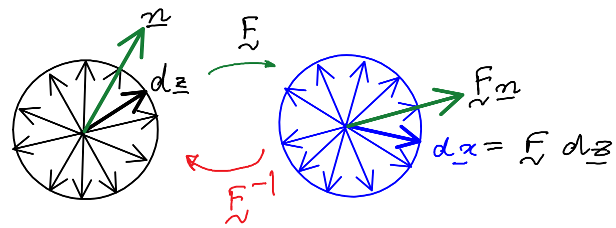
Lagrangian Strain in the direction $\underset{\sim}{n}$:
$$E = \tfrac{1}{2}(\lambda^2 - 1) = \tfrac{1}{2}(\underset{\sim}{n}\cdot\underset{\sim}{C}\underset{\sim}{n} - 1)$$

In general (for any direction):
$$\boxed{\underset{\sim}{E} = \tfrac{1}{2}(\underset{\sim}{C} - \underset{\sim}{I})} = \tfrac{1}{2}(\underset{\sim}{F}^T\underset{\sim}{F} - \underset{\sim}{I})$$

$\underset{\sim}{E}$: Green-Lagrange Strain Tensor.

Another interpretation

Since strain should be <u>zero</u> for a rigid body motion/deformation:



$\Rightarrow$ For any $d\underset{\sim}{z}$ vector:

$$d\underset{\sim}{z}\cdot d\underset{\sim}{z} = d\underset{\sim}{x}\cdot d\underset{\sim}{x} \qquad \forall\, d\underset{\sim}{z}$$
$$d\underset{\sim}{z}\cdot\underset{\sim}{I}\, d\underset{\sim}{z} = (\underset{\sim}{F}\, d\underset{\sim}{z})\cdot(\underset{\sim}{F}\, d\underset{\sim}{z}) = d\underset{\sim}{z}\cdot(\underset{\sim}{F}^T\underset{\sim}{F}\, d\underset{\sim}{z})$$
$$\Rightarrow d\underset{\sim}{z}\cdot(\underbrace{\underset{\sim}{F}^T\underset{\sim}{F}}_{\underset{\sim}{C}} - \underset{\sim}{I})\, d\underset{\sim}{z} = 0 \quad \Rightarrow \quad \boxed{\text{Strain} \propto (\underset{\sim}{C} - \underset{\sim}{I})} \qquad \underset{\sim}{E} = \tfrac{1}{2}(\underset{\sim}{C} - \underset{\sim}{I})$$

Alternatively:
$$(\underset{\sim}{F}^{-1} d\underset{\sim}{x})\cdot(\underset{\sim}{F}^{-1} d\underset{\sim}{x}) = d\underset{\sim}{x}\cdot d\underset{\sim}{x}$$
$$\Rightarrow \quad d\underset{\sim}{x}\cdot((\underset{\sim}{F}^{-1})^T\underset{\sim}{F}^{-1} d\underset{\sim}{x}) = d\underset{\sim}{x}\cdot\underset{\sim}{I}\, d\underset{\sim}{x}$$
$$\Rightarrow \quad d\underset{\sim}{x}\cdot((\underset{\sim}{F}\underset{\sim}{F}^T)^{-1} d\underset{\sim}{x}) = d\underset{\sim}{x}\cdot\underset{\sim}{I}\, d\underset{\sim}{x}$$
$$\Rightarrow \quad d\underset{\sim}{x}\cdot(\underset{\sim}{I} - \underset{\sim}{B}^{-1})\, d\underset{\sim}{x} = 0 \qquad \text{where} \qquad \underset{\sim}{B} = \underset{\sim}{F}\underset{\sim}{F}^T$$

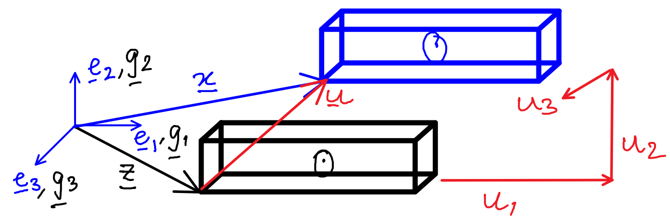
$\underset{\sim}{B}$: Left Cauchy-Green deformation Tensor
(or Almansi tensor; Finger Tensor)

Similarly
$$\boxed{\text{Strain} \propto \underset{\sim}{I} - \underset{\sim}{B}^{-1}} \qquad \underset{\sim}{H} = \tfrac{1}{2}(\underset{\sim}{I} - \underset{\sim}{B}^{-1})$$

Euler-Almansi Strain Tensor ($\underset{\sim}{e}$)

Examples:

(i) Translation $\underset{\sim}{x} = \underset{\sim}{\phi}(\underset{\sim}{z}) = \underset{\sim}{z} + \underset{\sim}{u}$

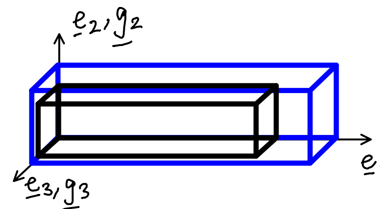


$\underset{\sim}{F} = \underset{\sim}{I}$; $\underset{\sim}{C} = \underset{\sim}{I}$ $\qquad \underset{\sim}{E} = \underset{\sim}{0}$

$(\lambda) \qquad (\lambda^2)$

(ii) Uniform Expansion in all 3 directions $\underset{\sim}{x} = \underset{\sim}{\phi}(\underset{\sim}{z}) = \alpha \underset{\sim}{z}$

$\underset{\sim}{F} = \alpha \underset{\sim}{I}$; $\underset{\sim}{C} = \alpha^2 \underset{\sim}{I}$; $\underset{\sim}{E} = \frac{1}{2}(\alpha^2 - 1)\underset{\sim}{I}$



Examples in the book:

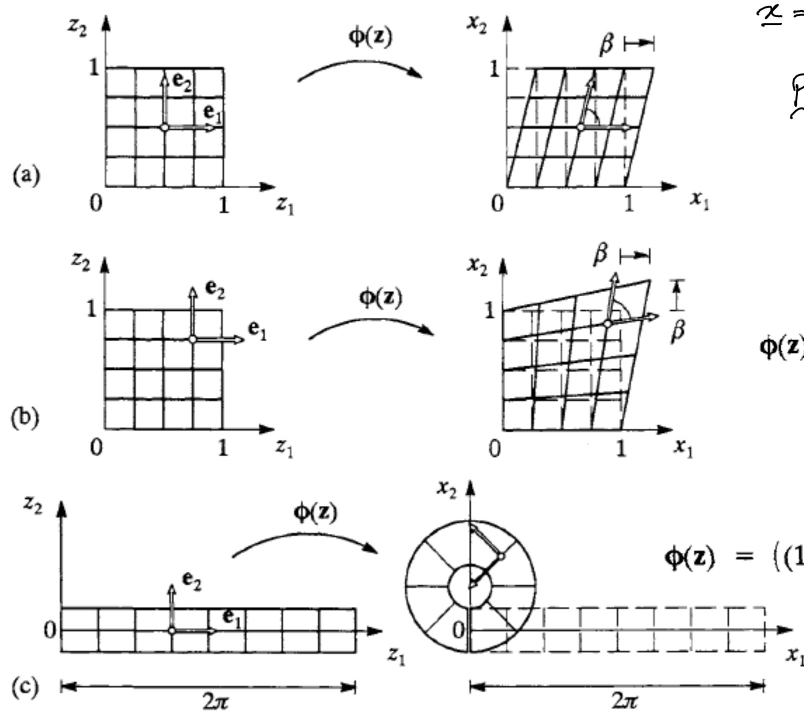


Figure 36 Shearing of base vectors for the example deformation maps (a) simple shear, (b) compound shearing and extension, and (c) pure bending

$\underset{\sim}{x} = \underset{\sim}{\phi}(\underset{\sim}{z}) = (z_1 + \beta z_2)\underset{\sim}{e_1} + z_2 \underset{\sim}{e_2} + z_3 \underset{\sim}{e_3}$

$$\underset{\sim}{F} \sim \begin{bmatrix} 1 & \beta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \underset{\sim}{C} \sim \begin{bmatrix} 1 & \beta & 0 \\ \beta & 1+\beta^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{E} \sim \frac{1}{2}\begin{bmatrix} 0 & \beta & 0 \\ \beta & \beta^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{\phi}(\mathbf{z}) = (z_1 + \beta z_1 z_2)\mathbf{e}_1 + (z_2 + \beta z_1 z_2)\mathbf{e}_2 + z_3 \mathbf{e}_3$$

$$\mathbf{F} \sim \begin{bmatrix} 1+\beta z_2 & \beta z_1 & 0 \\ \beta z_2 & 1+\beta z_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{\phi}(\mathbf{z}) = ((1-z_2)\sin z_1)\mathbf{e}_1 + (1-(1-z_2)\cos z_1)\mathbf{e}_2 + z_3\mathbf{e}_3$$

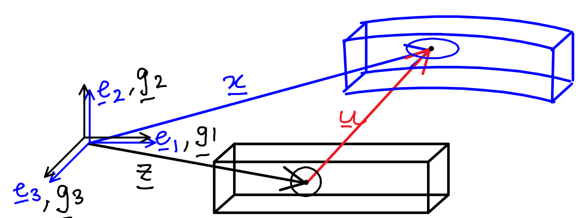
$$\mathbf{F} \sim \begin{bmatrix} (1-z_2)\cos z_1 & -\sin z_1 & 0 \\ (1-z_2)\sin z_1 & \cos z_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{C} \sim \begin{bmatrix} (1-z_2)^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Physical significance of components of *C* and *E*

Recall : $\lambda^2(\underset{\sim}{n}) = \underset{\sim}{n} \cdot \underset{\sim}{C}\underset{\sim}{n}$

and $\frac{1}{2}(\lambda^2(\underset{\sim}{n}) - 1) = \underset{\sim}{n} \cdot \underset{\sim}{E}\underset{\sim}{n}$



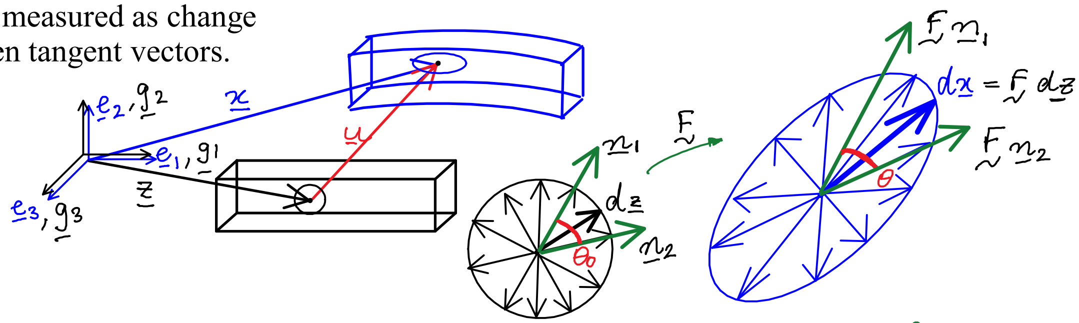
Note : $C_{ij} = \underset{\sim}{g_i} \cdot \underset{\sim}{C}\, \underset{\sim}{g_j}$

$E_{ij} = \underset{\sim}{g_i} \cdot \underset{\sim}{E}\, \underset{\sim}{g_j}$

Thus C_{11}, C_{22}, C_{33} represent "normal" stretches in $\underset{\sim}{g_1}, \underset{\sim}{g_2}, \underset{\sim}{g_3}$ directions.
(E_{11}, E_{22}, E_{33} represent "normal" strains in $\underset{\sim}{g_1}\, \underset{\sim}{g_2}\, \underset{\sim}{g_3}$ directions.)

Shearing components of *C* and *E*

Shear is usually measured as change in angles between tangent vectors.



Consider

$$(\underset{\sim}{F}\,\underset{\sim}{n}_1)\cdot(\underset{\sim}{F}\,\underset{\sim}{n}_2) = \|\underset{\sim}{F}\,\underset{\sim}{n}_1\|\;\|\underset{\sim}{F}\,\underset{\sim}{n}_2\|\;\cos\theta(\underset{\sim}{F}\,\underset{\sim}{n}_1,\underset{\sim}{F}\,\underset{\sim}{n}_2)$$

$$\Rightarrow \cos\theta(\underset{\sim}{F}\,\underset{\sim}{n}_1,\underset{\sim}{F}\,\underset{\sim}{n}_2) = \frac{(\underset{\sim}{F}\,\underset{\sim}{n}_1)\cdot(\underset{\sim}{F}\,\underset{\sim}{n}_2)}{\|\underset{\sim}{F}\,\underset{\sim}{n}_1\|\;\|\underset{\sim}{F}\,\underset{\sim}{n}_2\|} = \frac{\underset{\sim}{n}_1\cdot\overbrace{(\underset{\sim}{F}^T\underset{\sim}{F})}^{\underset{\sim}{C}}\,\underset{\sim}{n}_2}{\lambda(\underset{\sim}{n}_1)\;\lambda(\underset{\sim}{n}_2)}$$

Recall:

$$\lambda(\underset{\sim}{n}_1) = \frac{\|\underset{\sim}{F}\,\underset{\sim}{n}_1\|}{\|\underset{\sim}{n}_1\| \to 1}$$

$$\lambda(\underset{\sim}{n}_2) = \frac{\|\underset{\sim}{F}\,\underset{\sim}{n}_2\|}{\|\underset{\sim}{n}_2\| \to 1}$$

change in angle:

$$\theta_0(\underset{\sim}{n}_1,\underset{\sim}{n}_2) - \theta(\underset{\sim}{F}\,\underset{\sim}{n}_1,\underset{\sim}{F}\,\underset{\sim}{n}_2)$$

If $\underset{\sim}{n}_1 \perp \underset{\sim}{n}_2$ shear $= \pi/2 - \theta(\underset{\sim}{F}\,\underset{\sim}{n}_1,\underset{\sim}{F}\,\underset{\sim}{n}_2)$

If we choose $\underset{\sim}{n}_1$ and $\underset{\sim}{n}_2$ to be $\underset{\sim}{g}_i$ and $\underset{\sim}{g}_j$ basis vectors:

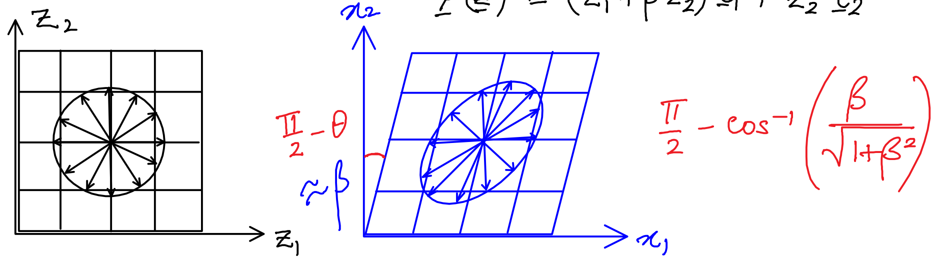
$$\text{Shear} = \pi/2 - \cos^{-1}\left(\frac{\underset{\sim}{g}_i\cdot\underset{\sim}{C}\,\underset{\sim}{g}_j}{\lambda(\underset{\sim}{g}_i)\,\lambda(\underset{\sim}{g}_j)}\right) = \frac{\pi}{2} - \cos^{-1}\left(\frac{C_{ij}}{\sqrt{C_{ii}}\,\sqrt{C_{jj}}}\right) \qquad (i,j:\text{ no sum})$$

Example:
(simple shear)

$$\underset{\sim}{F} \sim \begin{bmatrix} 1 & \beta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{C} \sim \begin{bmatrix} 1 & \beta & 0 \\ \beta & 1+\beta^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{\phi}(\underset{\sim}{Z}) = (Z_1+\beta Z_2)\,\underset{\sim}{e}_1 + Z_2\,\underset{\sim}{e}_2$$

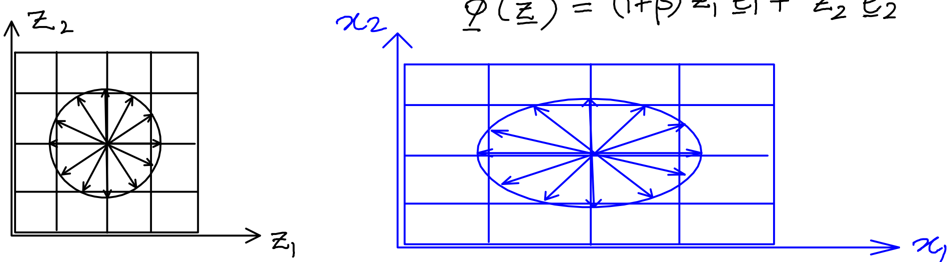


Example:
(simple extension)

$$\underset{\sim}{F} \sim \begin{bmatrix} 1+\beta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{C} \sim \begin{bmatrix} (1+\beta)^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

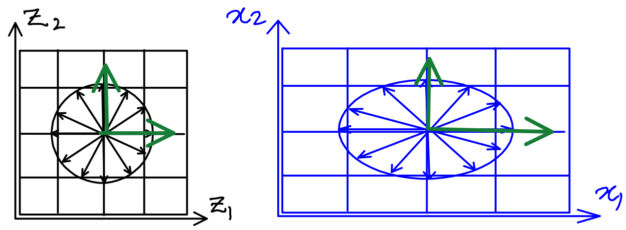
$$\underset{\sim}{\phi}(\underset{\sim}{Z}) = (1+\beta)Z_1\,\underset{\sim}{e}_1 + Z_2\,\underset{\sim}{e}_2$$



Note: The *zero* off-diagonal components of C in this case only mean that there is no shearing between the basis vectors $\boldsymbol{g}_1$, $\boldsymbol{g}_2$ and $\boldsymbol{g}_3$ of this particular coordinate system.
Clearly there are other pairs of vectors $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ for which there is definite shearing, even for this simple extension problem.

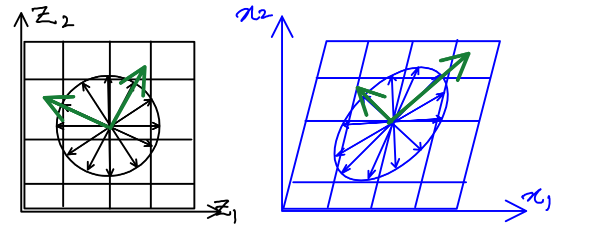
<u>Principal Deformations and Strains</u>

From the preceding discussion one can see that for any deformation, a small neighborhood of a point deforms in a way that there is both stretching and shearing.



i.e. a sphere of arbitrary infinitesimal undeformed tangent vectors d$\boldsymbol{z}$ is mapped to an ellipsoid of deformed tangent vectors d$\boldsymbol{x}$.

Thus there would be some directions in which the stretching is extremum (maximum / minimum).



However, unlike the effect of a symmetric tensor (where these extremal are not rotated), in this case, the extremal tangent vectors will in general have both stretching and rotation.

To find these directions of extremal stretch note: $\lambda^2(\underline{n}) = \underline{n} \cdot \underset{\sim}{C}\,\underline{n}$

To maximize/minimize $\lambda^2(\underline{n})$ subject to $\|\underline{n}\| = 1$ i.e. $\underline{n}\cdot\underline{n} = 1$

Consider the function $\mathcal{L}(\underline{n}, \mu) \equiv \lambda^2(\underline{n}) - \mu\,(\underline{n}\cdot\underline{n} - 1)$

Now, to extremize : $\dfrac{\partial \mathcal{L}}{\partial \underline{n}} = \underset{\sim}{C}\,\underline{n} - \mu\,\underline{n} = \underline{0} \Rightarrow \boxed{\underset{\sim}{C}\,\underline{n} = \mu\,\underline{n}}$

Eigenvalue Problem

$$\frac{\partial \mathcal{L}}{\partial \mu} = \underline{n}\cdot\underline{n} - 1 = 0 \Rightarrow \|\underline{n}\| = 1$$

Eigenvalues and Eigenvectors of $\boldsymbol{C}$ are found the same way as any symmetric tensor and have the same physical interpretations.

Note: stretches in the Eigenvector directions :

$$\lambda^2(\underline{n}_i) = \underline{n}_i \cdot \underset{\sim}{C}\,\underline{n}_i = \mu(\underline{n}_i\cdot\underline{n}_i) = \mu \Rightarrow \boxed{\lambda = \sqrt{\mu}}$$

(no sum)

$$\text{shear} = \frac{\pi}{2} - \cos^{-1}\left(\frac{\underline{n}_i\cdot\underset{\sim}{C}\,\underline{n}_j}{\lambda(\underline{n}_i)\,\lambda(\underline{n}_j)}\right) = \frac{\pi}{2} - \cos^{-1}\left(\frac{\mu_j\overbrace{(\underline{n}_i\cdot\underline{n}_j)}^{\delta_{ij}}}{\sqrt{\mu_i}\,\sqrt{\mu_j}}\right)$$

$\Rightarrow$ Shear between $\underline{n}_i$ and $\underline{n}_j$ (for $i \neq j$) $= 0$

Similarly principal values of the Lagrangian strain tensor: $\underset{\sim}{E}\,\underline{n} = \gamma\,\underline{n}$

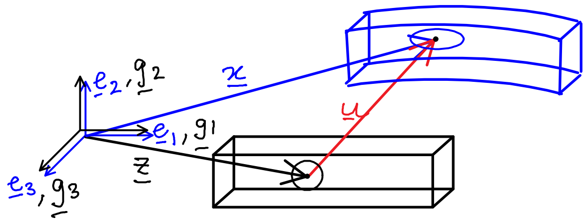
$$\Rightarrow \tfrac{1}{2}\left(\underset{\sim}{C} - \underset{\sim}{I}\right)\underline{n} = \gamma\,\underline{n} \quad \Rightarrow \quad \underset{\sim}{C}\,\underline{n} = \underbrace{(1-2\gamma)}_{\mu}\,\underline{n}$$

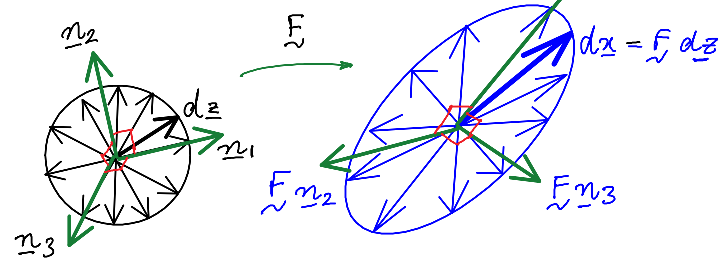
$$\Rightarrow \mu = 1 + 2\gamma \Rightarrow \gamma_i = \tfrac{1}{2}(\mu_i - 1)$$

$\Rightarrow \boxed{\gamma_i = \tfrac{1}{2}\left(\lambda_i^2 - 1\right)}$ and same Eigenvectors.

Rotation and Stretch (Polar Decomposition) $\boldsymbol{F} = \boldsymbol{R}\,\boldsymbol{U} = \boldsymbol{V}\,\boldsymbol{R}$

A very useful interpretation of the deformation gradient is that it causes simultaneous stretching and rotation of tangent vectors.

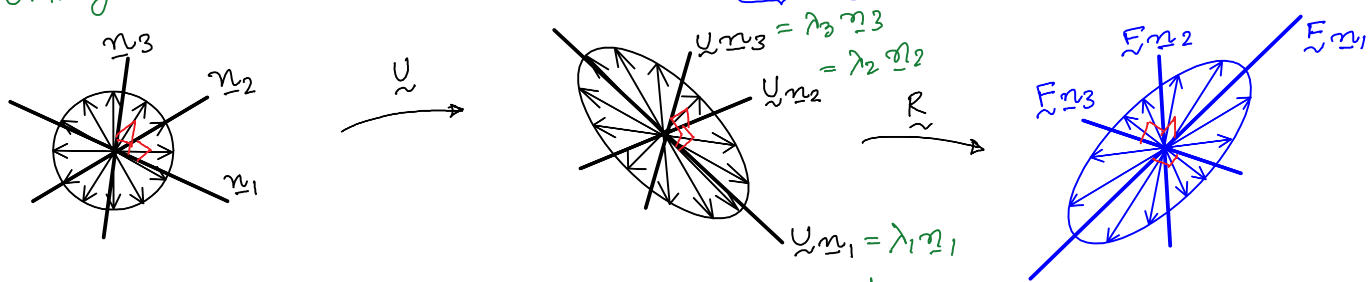




However one can also express the effect of $\boldsymbol{F}$ in terms of a sequence of stretching and rotation operations:

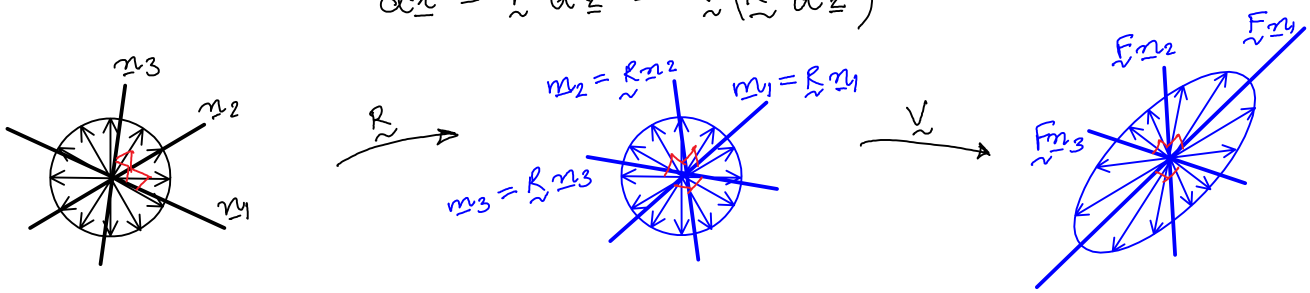
$\boldsymbol{F} = \boldsymbol{R}\,\boldsymbol{U}$ — $\boldsymbol{U}$ symmetric, $\boldsymbol{R}$ orthogonal

$$d\underline{x} = \underset{\sim}{F}\, d\underline{z} = \underset{\sim}{R}\left(\underset{\sim}{U}\, d\underline{z}\right)$$



Or a sequence of rotation and stretching operations: $\boldsymbol{F} = \boldsymbol{V}\,\boldsymbol{R}$ — $\boldsymbol{V}$ symmetric, $\boldsymbol{R}$ orthogonal

$$d\underline{x} = \underset{\sim}{F}\, d\underline{z} = \underset{\sim}{V}\left(\underset{\sim}{R}\, d\underline{z}\right)$$



Note: Left & Right Cauchy Green deformation tensors: $\underset{\sim}{B}$ and $\underset{\sim}{C}$

(capture only the stretching part of deformation, not rotation)

$$\underset{\sim}{B} = \underset{\sim}{F}\,\underset{\sim}{F}^T = (\underset{\sim}{V}\,\underset{\sim}{R})(\underset{\sim}{V}\,\underset{\sim}{R})^T = \underset{\sim}{V}\,(\underset{\sim}{R}\,\underset{\sim}{R}^T)\,\underset{\sim}{V}^T, \quad \underset{\sim}{R}\,\underset{\sim}{R}^T = \underset{\sim}{I}$$

$$\boxed{\underset{\sim}{B} = \underset{\sim}{V}\,\underset{\sim}{V}}$$

$$\underset{\sim}{C} = \underset{\sim}{F}^T\underset{\sim}{F}$$

$$\underset{\sim}{C} = (\underset{\sim}{R}\,\underset{\sim}{U})^T(\underset{\sim}{R}\,\underset{\sim}{U})$$

$$\underset{\sim}{C} = \underset{\sim}{U}^T(\underset{\sim}{R}^T\underset{\sim}{R})\,\underset{\sim}{U}, \quad \underset{\sim}{R}^T\underset{\sim}{R} = \underset{\sim}{I}$$

$$\boxed{\underset{\sim}{C} = \underset{\sim}{U}\,\underset{\sim}{U}}$$

Spectral decomposition of $\boldsymbol{B}$ and $\boldsymbol{C}$

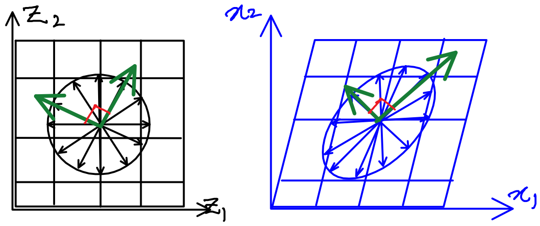
(to find $\boldsymbol{V}$ and $\boldsymbol{U}$ and $\boldsymbol{R}$)

$$\underset{\sim}{B} = \sum_{i=1}^{3} \lambda_i^2\, \underset{\sim}{m}_i \otimes \underset{\sim}{m}_i \qquad \underset{\sim}{C} = \sum_{i=1}^{3} \lambda_i^2\, \underset{\sim}{n}_i \otimes \underset{\sim}{n}_i$$

$$\underset{\sim}{V} = \sum_{i=1}^{3} \lambda_i\, \underset{\sim}{m}_i \otimes \underset{\sim}{m}_i \qquad \underset{\sim}{U} = \sum_{i=1}^{3} \lambda_i\, \underset{\sim}{n}_i \otimes \underset{\sim}{n}_i$$

$$\underset{\sim}{F} = \underset{\sim}{V}\,\underset{\sim}{R} \Rightarrow \underset{\sim}{R} = \underset{\sim}{V}^{-1}\underset{\sim}{F} \quad ; \quad \underset{\sim}{F} = \underset{\sim}{R}\,\underset{\sim}{U} \Rightarrow \underset{\sim}{R} = \underset{\sim}{F}\,\underset{\sim}{U}^{-1}$$

Example: (simple shear)



$$\phi(\underset{\sim}{z}) = (z_1 + 0.2 z_2)\, \underset{\sim}{e}_1 + z_2\, \underset{\sim}{e}_2$$

$$\underset{\sim}{F} \sim \begin{bmatrix} 1 & 0.2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \underset{\sim}{C} \sim \begin{bmatrix} 1 & 0.2 & 0 \\ 0.2 & 1.04 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

```
>> [N, D] = eig(C)
N =
  -0.7415         0    0.6710
   0.6710         0    0.7415
        0    1.0000         0
D =
   0.8190         0         0
        0    1.0000         0
        0         0    1.2210
```

(columns of N: n_2, n_3, n_1; diagonal of D: λ_2^2, λ_3^2, λ_1^2)

```
>> L = sqrt(D)
L =
   0.9050         0         0
        0    1.0000         0
        0         0    1.1050
>> U = N*L*inv(N)
U =
   0.9950    0.0995         0
   0.0995    1.0149         0
        0         0    1.0000
```

(diagonal of L: λ_2, λ_3, λ_1)

$$\underset{\sim}{U}\, \underset{\sim}{n}_i = \lambda_i\, \underset{\sim}{n}_i$$

$$[\underset{\sim}{U}][\underset{\sim}{N}] = [\underset{\sim}{N}][\underset{\sim}{L}]$$

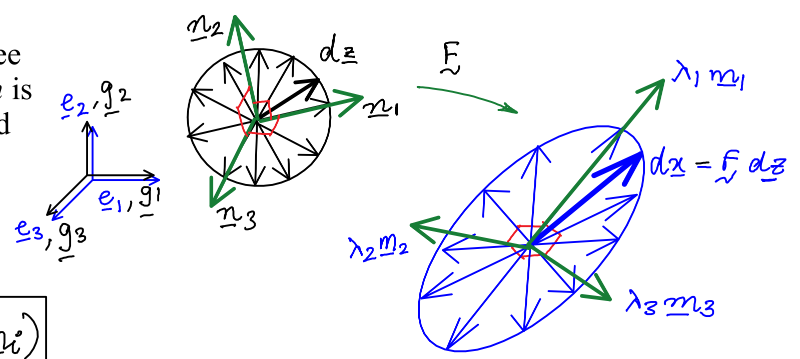
$$[\underset{\sim}{U}] = [\underset{\sim}{N}][\underset{\sim}{L}][\underset{\sim}{N}]^{-1}$$

```
>> R = F*inv(U)
R =
   0.9950    0.0995         0
  -0.0995    0.9950         0
        0         0    1.0000
>> R'*R
ans =
   1.0000    0.0000         0
   0.0000    1.0000         0
        0         0    1.0000
>> R*R'
ans =
   1.0000    0.0000         0
   0.0000    1.0000         0
        0         0    1.0000
```

$\underset{\sim}{R}^T \underset{\sim}{R} = \underset{\sim}{I}$, $\quad \underset{\sim}{R}\, \underset{\sim}{R}^T = \underset{\sim}{I}$

(Psuedo-) Spectral Decomposition of *F*

From the preceding discussion we can see that the effect of ***F*** on all Eigenvectors ***n*** is to stretch them (by varying amounts) and rotate them all by the same amount.



$$\Rightarrow \underset{\sim}{F}\, \underset{\sim}{n}_i = \lambda_i\, \underset{\sim}{m}_i \quad (i\text{: no sum})$$

This leads to: $\boxed{\underset{\sim}{F} = \sum_{i=1}^{3} \lambda_i (\underset{\sim}{m}_i \otimes \underset{\sim}{n}_i)}$

$$\text{Verify: } \underset{\sim}{F}\, \underset{\sim}{u} = \sum_{i=1}^{3} \lambda_i (\underset{\sim}{m}_i \otimes \underset{\sim}{n}_i)\Big(\sum_{j=1}^{3} u_j\, \underset{\sim}{n}_j\Big) = \sum_{i=1}^{3} \lambda_i\, u_i\, \underset{\sim}{m}_i = \underset{\sim}{v} = \underset{\sim}{F}\, \underset{\sim}{u}$$

(δ_{ij})

Further, this helps us express the rotation tensor ***R*** as: $\boxed{\underset{\sim}{R} = \sum_{i=1}^{3} \underset{\sim}{m}_i \otimes \underset{\sim}{n}_i} = \sum_{i=1}^{3} \frac{1}{\lambda_i} \underbrace{(\underset{\sim}{F}\, \underset{\sim}{n}_i)}_{\lambda_i m_i} \otimes \underset{\sim}{n}_i$

Physical interpretation of **Principal** invariants of *U* and *C*:

$$I_U = \lambda_1 + \lambda_2 + \lambda_3 \quad \text{(length)}$$

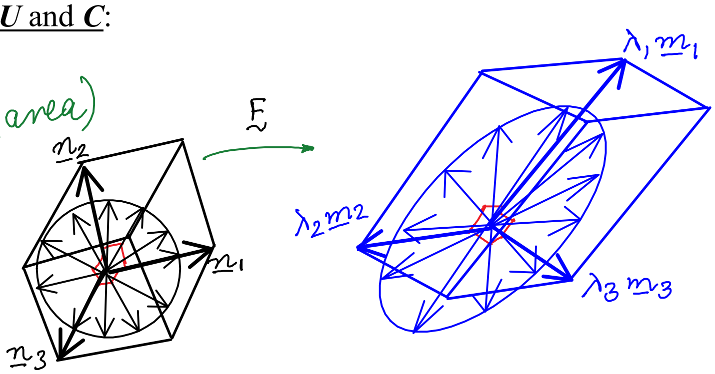
$$II_U = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 \quad \text{(area)}$$

$$III_U = \lambda_1\lambda_2\lambda_3 \quad \text{(volume)}$$

$$I_C = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$

$$II_C = (\lambda_1\lambda_2)^2 + (\lambda_1\lambda_2)^2 + (\lambda_2\lambda_3)^2$$

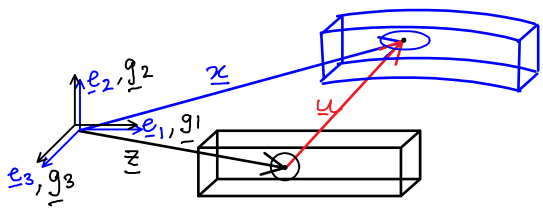
$$III_C = (\lambda_1\lambda_2\lambda_3)^2$$

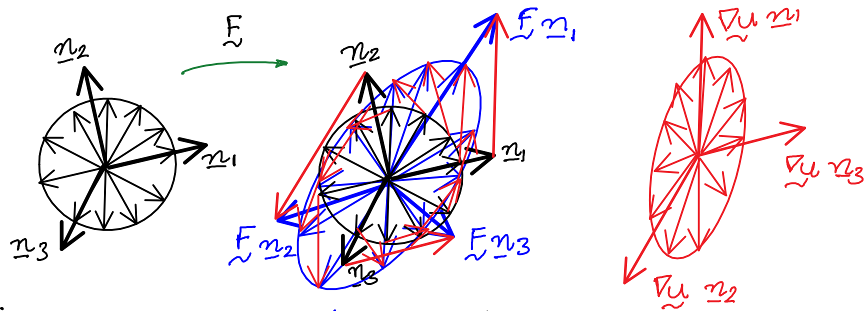


Deformation Gradient ($\boldsymbol{F}$) and Displacement Gradient ($\boldsymbol{\nabla u}$)

Recall: $\underset{\sim}{\phi}(\underset{\sim}{z}) = \underset{\sim}{z} + \underset{\sim}{u}(\underset{\sim}{z})$

Deformation gradient: $\underset{\sim}{F} = \nabla_{\underset{\sim}{z}}\, \underset{\sim}{\phi}(\underset{\sim}{z}) = \underbrace{\nabla_{\underset{\sim}{z}}(\underset{\sim}{z})}_{\underset{\sim}{I}} + \underbrace{\nabla_{\underset{\sim}{z}}\, \underset{\sim}{u}(\underset{\sim}{z})}_{\text{Displacement gradient}} \Rightarrow \boxed{\underset{\sim}{F} = \underset{\sim}{I} + \nabla \underset{\sim}{u}}$





Right Cauchy-Green Deformation Tensor

$d\underset{\sim}{x} = \underset{\sim}{F}\, d\underset{\sim}{z}$

$d\underset{\sim}{x} = d\underset{\sim}{z} + \nabla \underset{\sim}{u}\, d\underset{\sim}{z}$

$$\underset{\sim}{C} = \underset{\sim}{F}^T \underset{\sim}{F} = (\underset{\sim}{I}^T + \nabla \underset{\sim}{u}^T)(\underset{\sim}{I} + \nabla \underset{\sim}{u})$$

$$\Rightarrow \quad \underset{\sim}{C} = \underset{\sim}{I} + \nabla \underset{\sim}{u}^T + \nabla \underset{\sim}{u} + (\nabla \underset{\sim}{u}^T \nabla \underset{\sim}{u})$$

$$\nabla \underset{\sim}{u} = \frac{\partial u_i}{\partial z_j} \cdot (\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j)$$

Green Lagrange Strain Tensor

$$\underset{\sim}{E} = \frac{1}{2}(\underset{\sim}{C} - \underset{\sim}{I}) = \frac{1}{2}\left(\nabla \underset{\sim}{u}^T + \nabla \underset{\sim}{u} + \boxed{\nabla \underset{\sim}{u}^T \nabla \underset{\sim}{u}}\right)$$

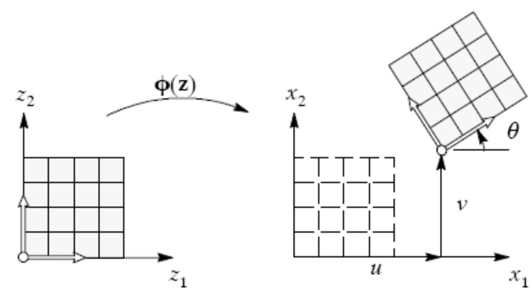
strain ($\underset{\sim}{E}$) is a non-linear function of $\underset{\sim}{u}(\underset{\sim}{z})$

Linearized Strain:

$$\underset{\sim}{\epsilon} = \frac{1}{2}(\nabla \underset{\sim}{u}^T + \nabla \underset{\sim}{u})$$

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

Example
(Ref: Pg 76, Hjelmstad)



$$\boldsymbol{\phi}(\mathbf{z}) = (u + z_1\cos\theta - z_2\sin\theta)\mathbf{e}_1 + (v + z_1\sin\theta + z_2\cos\theta)\mathbf{e}_2 + z_3\mathbf{e}_3$$

$$\underset{\sim}{\phi}(\underset{\sim}{z}) = \underset{\sim}{z} + \underbrace{[u + z_1(\cos\theta - 1) - z_2\sin\theta]}_{u_1(z_1, z_2, z_3)}\underset{\sim}{e}_1 + \underbrace{[v + z_1\sin\theta + z_2(\cos\theta - 1)]}_{u_2(z_1, z_2, z_3)}\underset{\sim}{e}_2 + \underbrace{0\, \underset{\sim}{e}_3}_{u_3(z_1, z_2, z_3) = 0}$$

$$\nabla \underset{\sim}{u} \sim \begin{bmatrix} (\cos\theta - 1) & -\sin\theta & 0 \\ \sin\theta & (\cos\theta - 1) & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \underset{\sim}{F} = \underset{\sim}{I} + \nabla \underset{\sim}{u} \sim \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \underset{\sim}{C} = \underset{\sim}{F}^T \underset{\sim}{F} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \underset{\sim}{I} \quad \Rightarrow \underset{\sim}{E} = \underset{\sim}{0} \quad \text{(Rigid Body Motion)}$$

However:

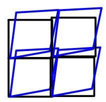
$$\underset{\sim}{\epsilon} = \frac{1}{2}(\nabla \underset{\sim}{u}^T + \nabla \underset{\sim}{u}) \sim \begin{bmatrix} \cos\theta - 1 & 0 & 0 \\ 0 & \cos\theta - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \approx \begin{bmatrix} -\theta^2/2 & 0 & 0 \\ 0 & -\theta^2/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \approx \underset{\sim}{0} \quad \text{(only if } \theta \approx 0\text{)}$$

Compatibility of Strains

Meaning of compatibility

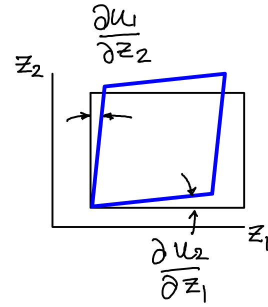
- Given $\underline{\phi}(\underline{z}) = \underline{z} + \underline{u}(\underline{z}) \rightsquigarrow \underset{\sim}{\epsilon}(\underline{u})$ (Automatically satisfied)
- Given $\underset{\sim}{\epsilon}(\underline{z})$ or $\underset{\sim}{E}(\underline{z}) \rightsquigarrow \underline{u}(\underline{z})$?

$\underline{u}(\underline{z})$



?

$$\underline{u}(\underline{z}) \rightsquigarrow \underset{\sim}{\epsilon} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix}$$



For linearized (Small strain):

Note
$$\epsilon_{11} = \frac{\partial u_1}{\partial z_1} \Rightarrow \frac{\partial^2 \epsilon_{11}}{\partial z_2^2} = \frac{\partial^3 u_1}{\partial z_1 \partial z_2^2}$$

Shear $\approx \left(\frac{\pi}{2} - \theta\right) \approx \left(\frac{\partial u_1}{\partial z_2} + \frac{\partial u_2}{\partial z_1}\right) \approx 2\epsilon_{12}$

$$\epsilon_{22} = \frac{\partial u_2}{\partial z_2} \Rightarrow \frac{\partial^2 \epsilon_{22}}{\partial z_1^2} = \frac{\partial^3 u_2}{\partial z_1^2 \partial z_2}$$

and
$$2\epsilon_{12} = \frac{\partial u_1}{\partial z_2} + \frac{\partial u_2}{\partial z_1} \Rightarrow 2\frac{\partial^2 \epsilon_{12}}{\partial z_1 \partial z_2} = \frac{\partial^3 u_1}{\partial z_1 \partial z_2^2} + \frac{\partial^3 u_2}{\partial z_1^2 \partial z_2}$$

$$\Rightarrow \boxed{\frac{\partial^2 \epsilon_{11}}{\partial z_2^2} + \frac{\partial^2 \epsilon_{22}}{\partial z_1^2} = 2\frac{\partial^2 \epsilon_{12}}{\partial z_1 \partial z_2}}$$

(similarly 2 more equations)

In addition:

$$\epsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial z_2} + \frac{\partial u_2}{\partial z_1}\right) \Rightarrow \frac{\partial \epsilon_{12}}{\partial z_3} = \frac{1}{2}\left(\frac{\partial^2 u_1}{\partial z_2 \partial z_3} + \frac{\partial^2 u_2}{\partial z_1 \partial z_3}\right) \quad \ominus$$

Similarly
$$\frac{\partial \epsilon_{23}}{\partial z_1} = \frac{1}{2}\left(\frac{\partial^2 u_2}{\partial z_1 \partial z_3} + \frac{\partial^2 u_3}{\partial z_1 \partial z_2}\right) \quad \oplus$$

$$\frac{\partial \epsilon_{13}}{\partial z_2} = \frac{1}{2}\left(\frac{\partial^2 u_1}{\partial z_2 \partial z_3} + \frac{\partial^2 u_3}{\partial z_1 \partial z_2}\right) \quad \oplus$$

$$\boxed{\frac{\partial}{\partial z_3}\left(-\frac{\partial \epsilon_{12}}{\partial z_3} + \frac{\partial \epsilon_{23}}{\partial z_1} + \frac{\partial \epsilon_{13}}{\partial z_2}\right) = \frac{\partial}{\partial z_3}\left(\frac{\partial^2 u_3}{\partial z_1 \partial z_2}\right) = \frac{\partial^2 \epsilon_{33}}{\partial z_1 \partial z_2}}$$

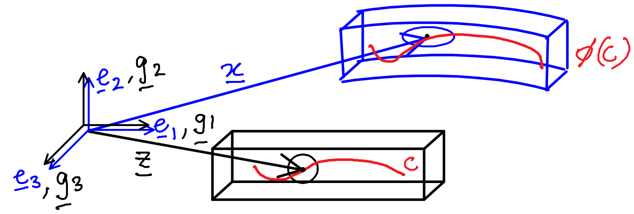
(similarly 2 more equations)

=> Total <u>6 equations of compatibility:</u>

$$\underline{\nabla} \times \underset{\sim}{\epsilon} \times \underline{\nabla} = \underset{\sim}{0}$$

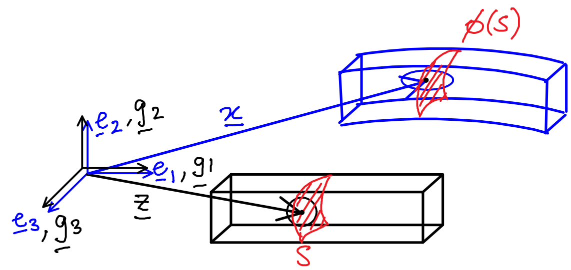
Local and Global Changes in Area and Volume

We have considered local changes in length using the stretch $\lambda(\underset{\sim}{z})$ at specific points along curves, and to find global change in length of some segment of a curve, we integrate the local stretch $\lambda(\underset{\sim}{z})$ along the curve.



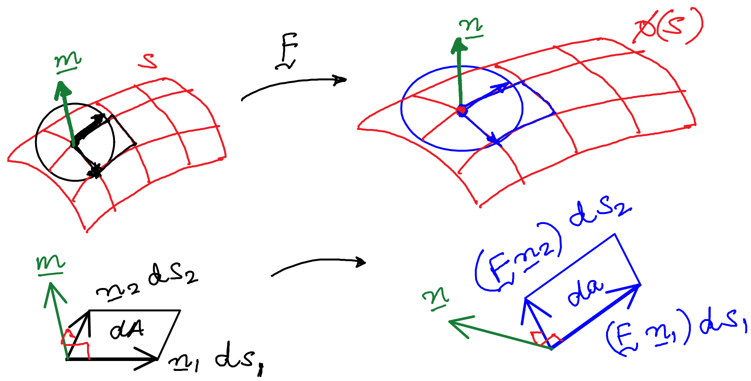
In the same way we can find local changes in area and volume of a body (at a specific point) and integrate that to find global/total changes in area and volume of a specific surface or volume.

$$\ell(\phi(C)) = \int_a^b \lambda(\underset{\sim}{z}(s))\, ds$$



Local change in Area:

Area is obtained by cross-product of 2 tangent vectors:



Note:

$$\underset{\sim}{m} = \frac{(\underset{\sim}{n}_1 \times \underset{\sim}{n}_1)}{\|\underset{\sim}{n}_1 \times \underset{\sim}{n}_2\|}$$

$$\underset{\sim}{n} = \frac{(\underset{\sim}{F}\underset{\sim}{n}_1) \times (\underset{\sim}{F}\underset{\sim}{n}_2)}{\|\underset{\sim}{F}\underset{\sim}{n}_1 \times \underset{\sim}{F}\underset{\sim}{n}_2\|}$$

Original Local Area:

$$dA = \|(\underset{\sim}{n}_1 \times \underset{\sim}{n}_2)\|\, ds_1\, ds_2$$

Note:

$$\underset{\sim}{m}\, dA = (\underset{\sim}{n}_1 \times \underset{\sim}{n}_2)\, ds_1\, ds_2$$

Deformed Local Area:

$$da = \|(\underset{\sim}{F}\underset{\sim}{n}_1) \times (\underset{\sim}{F}\underset{\sim}{n}_2)\|\, ds_1\, ds_2$$

$$\underset{\sim}{n}\, da = (\underset{\sim}{F}\underset{\sim}{n}_1 \times \underset{\sim}{F}\underset{\sim}{n}_2)\, ds_1\, ds_2$$

$$\underset{\sim}{F}^T \underset{\sim}{n}\, da = \underbrace{\underset{\sim}{F}^T (\underset{\sim}{F}\underset{\sim}{n}_1 \times \underset{\sim}{F}\underset{\sim}{n}_2)}_{\det(\underset{\sim}{F})(\underset{\sim}{n}_1 \times \underset{\sim}{n}_2)}\, ds_1\, ds_2 \rightarrow \text{Theorem 147(b) (Hjelmstad)}$$

$$\Rightarrow \underset{\sim}{n}\, da = \underset{\sim}{F}^{-T} \det(\underset{\sim}{F})\, \underbrace{(\underset{\sim}{n}_1 \times \underset{\sim}{n}_2)\, ds_1\, ds_2}_{\underset{\sim}{m}\, dA}$$

$$\Rightarrow \boxed{\underset{\sim}{n}\, da = \det(\underset{\sim}{F})\, \underset{\sim}{F}^{-T} \underset{\sim}{m}\, dA}$$ (Nanson's Formula) (Piola Transformation)

Ratio of local area change:

$$\frac{da}{dA} = \frac{\|\underset{\sim}{F}\underset{\sim}{n}_1 \times \underset{\sim}{F}\underset{\sim}{n}_2\|}{\|\underset{\sim}{n}_1 \times \underset{\sim}{n}_2\|} = \frac{\|\underset{\sim}{n}\, da\|}{\|\underset{\sim}{m}\, dA\|} = \boxed{\det(\underset{\sim}{F})\, \|\underset{\sim}{F}^{-T}\underset{\sim}{m}\|}$$

Original Total Area:

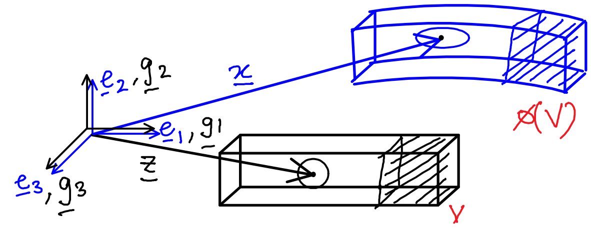
$$A = \iint_S dA$$

Deformed Total Area:

$$a = \iint_{\phi(S)} da = \iint_S \det(\underset{\sim}{F})\, \|\underset{\sim}{F}^{-T}\underset{\sim}{m}\|\, dA$$

Local change in Volume:

Volume is given by scalar triple product of 3 tangent vectors:



Original Local Volume:

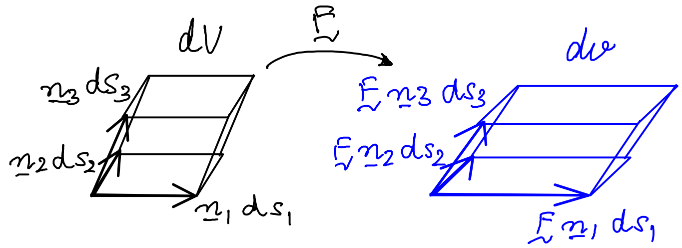
$$dV = (n_1 \times n_2) \cdot n_3 \; ds_1 \, ds_2 \, ds_3$$

Deformed Local Volume:

$$dv = \underbrace{(F n_1 \times F n_2) \cdot F n_3}_{\det(F)\,(n_1 \times n_2) \cdot n_3} \; ds_1 \, ds_2 \, ds_3$$

(Theorem 147(a) Hjelmstad)

$$\Rightarrow \boxed{dv = \det(F)\, dV}$$



Original total volume:

$$V = \iiint_V dV$$

Deformed Total Volume:

$$v = \iiint_{\phi(V)} dv = \iiint_V \det(F)\, dV$$

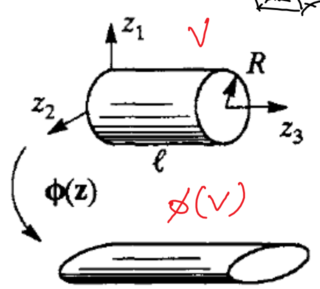
Example: Look at examples 16 and 17 in the textbook.

$dV = dr \, r d\theta \, dz_3$

78. A circular cylinder of length ℓ and radius R experiences the deformation characterized by the following map:

$$\phi(\mathbf{z}) = \alpha z_1 \mathbf{e}_1 + \beta z_2 \mathbf{e}_2 + \gamma z_3 \mathbf{e}_3$$

where α, β, and γ are constants of the motion. Find the volume of the deformed cylinder. Find the total surface area of the deformed cylinder. Find the principal stretches of the motion. What are the limits on the constants α, β, and γ?



$$F \sim \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix}$$

$$\det(F) = \alpha\beta\gamma$$

a) Volume of deformed cylinder: $v = \iiint_{\phi(V)} dv = \iiint_V \det(F)\, dV$

$$v = \iiint_V \alpha\beta\gamma \, dV = \alpha\beta\gamma \iiint_V dV = \alpha\beta\gamma\,(\pi R^2 \ell)$$

b) Surface Area: $a = \iint_{\phi(A)} da = \iint_A \det(F)\, \| F^{-T} m \| \, dA$

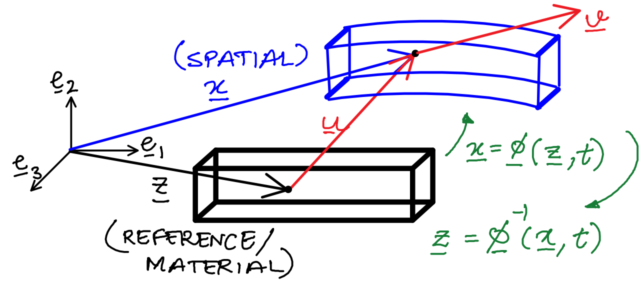
$$\left\{ m \sim \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} \right\}$$

Note $F^{-T} \sim \begin{bmatrix} 1/\alpha & 0 & 0 \\ 0 & 1/\beta & 0 \\ 0 & 0 & 1/\gamma \end{bmatrix} \Rightarrow \underset{\text{(end faces)}}{F^{-T}(\pm e_3)} \sim \pm \begin{bmatrix} 0 \\ 0 \\ 1/\gamma \end{bmatrix}$ & $\underset{\text{(curved face)}}{F^{-T} m} \sim \begin{bmatrix} \frac{\cos\theta}{\alpha} \\ \frac{\sin\theta}{\beta} \\ 0 \end{bmatrix}$

$$\Rightarrow a = 2 \iint_{\text{End faces}} \alpha\beta\gamma \left(\frac{1}{\gamma}\right) dA + \iint_{\text{Curved faces}} \alpha\beta\gamma \sqrt{\frac{\cos^2\theta}{\alpha^2} + \frac{\sin^2\theta}{\beta^2}} \cdot dA$$

Time dependent motion

All motion (and deformation) is time-dependent.
All the quantities we have defined thus far are for a particular instant of time t.



Recall:

$\boldsymbol{u} = \boldsymbol{u}_L(\boldsymbol{z},t) = \boldsymbol{x}(\boldsymbol{z},t) - \boldsymbol{z} = \boldsymbol{\phi}(\boldsymbol{z},t) - \boldsymbol{z}$ (Lagrangian)

$\boldsymbol{u} = \boldsymbol{u}_E(\boldsymbol{x},t) = \boldsymbol{x} - \boldsymbol{z}(\boldsymbol{x},t) = \boldsymbol{x} - \boldsymbol{\phi}^{-1}(\boldsymbol{x},t)$ (Eulerian)

Lagrangian / Reference / Material

Velocity $\underline{v}$

$$\underline{v}_L(\underline{z},t) \equiv \dot{\underline{x}}(\underline{z},t) = \frac{d\underline{x}}{dt}(\underline{z},t) = \boxed{\frac{\partial \underline{x}}{\partial t}} + \overbrace{\frac{\partial \underline{x}}{\partial \underline{z}}}^{F} \cdot \cancel{\frac{\partial \underline{z}}{\partial t}}^{\,0}$$

Acceleration $\underline{a}$

$$\underline{a}_L(\underline{z},t) \equiv \dot{\underline{v}}_L(\underline{z},t) = \frac{d^2\underline{x}}{dt^2}(\underline{z},t) = \boxed{\frac{\partial^2 \underline{x}}{\partial t^2}}$$

Eulerian / Spatial

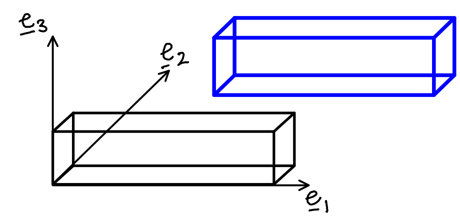
Velocity $\underline{v}$

$$\underline{v}_E(\underline{x},t) \equiv \dot{\underline{x}} = \frac{d\underline{x}}{dt} = \boxed{\frac{\partial \underline{x}}{\partial t}} = \underline{v}_L(\underline{z},t)$$

Acceleration $\underline{a}$

$$\underline{a}_E(\underline{x},t) \equiv \dot{\underline{v}}_E(\underline{x},t) = \frac{\partial \underline{v}_E}{\partial t} + \underbrace{\frac{\partial \underline{v}_E}{\partial \underline{x}}}_{\underline{L}\,:\ \text{velocity gradient}} \cdot \overbrace{\frac{\partial \underline{x}}{\partial t}}^{\underline{v}_E}$$

Example:



Let $\underline{x}(\underline{z},t) = \underline{\phi}(\underline{z},t) = (1+\alpha t^2)\underline{z}$

$\Rightarrow \underline{z}(\underline{x},t) = \underline{\phi}^{-1}(\underline{x},t) = \dfrac{\underline{x}}{(1+\alpha t^2)}$

$$\underline{v} = \underline{v}_L(\underline{z},t) = 2t\alpha\,\underline{z}$$

$$\underline{a} = \underline{a}_L(\underline{z},t) = 2\alpha\,\underline{z}$$

$$\underline{v} = \underline{v}_E(\underline{x},t) = \dot{\underline{x}} = \frac{2t\alpha\,\underline{x}}{(1+\alpha t^2)} \qquad \{= 2t\alpha\,\underline{z}\}$$

$$\underline{a} = \underline{a}_E(\underline{x},t) = \frac{\partial \underline{v}_E}{\partial t} + \frac{\partial \underline{v}_E}{\partial \underline{x}} \cdot \dot{\underline{x}}$$

$$\Rightarrow \underline{a} = \left(\frac{2\alpha\,\underline{x}}{(1+\alpha t^2)} - \cancel{\frac{2\alpha t\,\underline{x}}{(1+\alpha t^2)^2} \cdot 2\alpha t}\right) + \overbrace{\frac{2t\alpha\,\underline{I}}{(1+\alpha t^2)}}^{\underline{L}} \cancel{\frac{2t\alpha\,\underline{x}}{(1+\alpha t^2)}}$$

$$\Rightarrow \underline{a} = \frac{2\alpha\,\underline{x}}{(1+\alpha t^2)} \qquad \{= 2\alpha\,\underline{z}\}$$

Material time derivatives (for Eulerian descriptions)

For any scalar, vector or tensor:

- $\dfrac{d\rho}{dt}(\underline{x},t) = \dfrac{\partial \rho}{\partial t} + \dfrac{\partial \rho}{\partial \underline{x}} \cdot \underbrace{\boxed{\dfrac{\partial \underline{x}}{\partial t}}}_{\underline{v}}$

- $\dfrac{d}{dt}\underline{b}(\underline{x},t) = \dfrac{\partial \underline{b}}{\partial t} + \dfrac{\partial \underline{b}}{\partial \underline{x}} \underbrace{\boxed{\dfrac{\partial \underline{x}}{\partial t}}}_{\underline{v}}$

- $\dfrac{d}{dt}\underset{\sim}{T}(\underline{x},t) = \dfrac{\partial \underset{\sim}{T}}{\partial t} + \dfrac{\partial \underset{\sim}{T}}{\partial \underline{x}} \underbrace{\boxed{\dfrac{\partial \underline{x}}{\partial t}}}_{\underline{v}}$

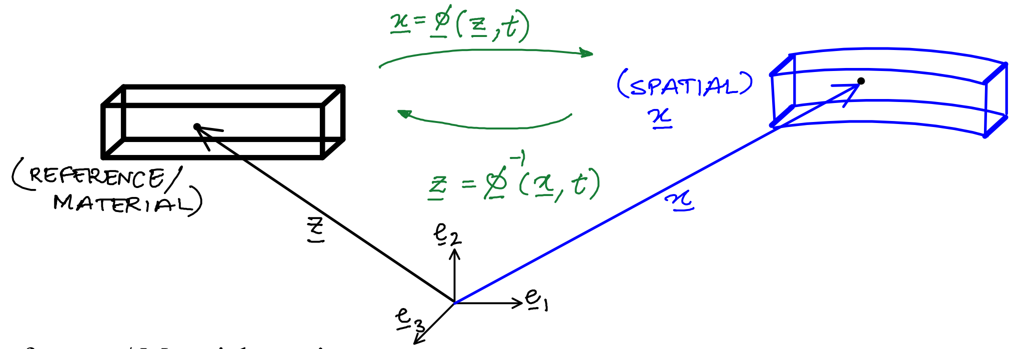
Note:

$$\frac{\partial \rho}{\partial \underline{x}} = \frac{\partial \rho}{\partial x_i}\,\underline{e}_i$$

$$\frac{\partial \underline{b}}{\partial \underline{x}} = \frac{\partial b_i}{\partial x_j}\,\underline{e}_i \otimes \underline{e}_j$$

$$\frac{\partial \underset{\sim}{T}}{\partial \underline{x}} = \frac{\partial T_{ij}}{\partial x_k}\,\underline{e}_i \otimes \underline{e}_j \otimes \underline{e}_k$$

Rate of change of deformations and strains



Lagrangian / Reference / Material

$$\dot{\underset{\sim}{F}} = \frac{d}{dt}\left(\frac{\partial \underset{\sim}{x}}{\partial \underset{\sim}{z}}\right) = \frac{\partial}{\partial \underset{\sim}{z}}\left(\frac{d\underset{\sim}{x}}{dt}\right) = \boxed{\frac{\partial \underset{\sim}{v}}{\partial \underset{\sim}{z}}} \quad (\dot{\underset{\sim}{F}})$$

$$\dot{\underset{\sim}{C}} = \frac{d}{dt}\left(\underset{\sim}{F}^T \underset{\sim}{F}\right) = \left(\dot{\underset{\sim}{F}}^T \underset{\sim}{F} + \underset{\sim}{F}^T \dot{\underset{\sim}{F}}\right)$$

$$\dot{\underset{\sim}{E}} = \frac{1}{2}\dot{\underset{\sim}{C}}$$

Eulerian / Spatial

$$\dot{\underset{\sim}{F}} = \underbrace{\frac{\partial \underset{\sim}{v}}{\partial \underset{\sim}{x}}}_{\underset{\sim}{L}} \underbrace{\frac{\partial \underset{\sim}{x}}{\partial \underset{\sim}{z}}}_{\underset{\sim}{F}} = \underset{\sim}{L}\,\underset{\sim}{F} \Rightarrow \boxed{\underset{\sim}{L} = \dot{\underset{\sim}{F}}\,\underset{\sim}{F}^{-1}}$$

$$\underset{\sim}{D} = \frac{1}{2}\left(\underset{\sim}{L} + \underset{\sim}{L}^T\right)$$ Symmetric Rate of Deformation

$$\underset{\sim}{W} = \frac{1}{2}\left(\underset{\sim}{L} - \underset{\sim}{L}^T\right)$$ Skew-symmetric Spin tensor

Note:
$$\underset{\sim}{D} = \frac{1}{2}\left(\dot{\underset{\sim}{F}}\underset{\sim}{F}^{-1} + \underset{\sim}{F}^{-T}\dot{\underset{\sim}{F}}^T\right)$$
$$= \frac{1}{2}\left(\left(\dot{\underset{\sim}{R}}\underset{\sim}{U} + \underset{\sim}{R}\dot{\underset{\sim}{U}}\right)\left(\underset{\sim}{U}^{-1}\underset{\sim}{R}^{-1}\right) + \left(\underset{\sim}{R}\,\underset{\sim}{U}^{-1}\right)\left(\underset{\sim}{U}\dot{\underset{\sim}{R}}^T + \dot{\underset{\sim}{U}}\underset{\sim}{R}^T\right)\right)$$
$$= \frac{1}{2}\left(\cancel{\dot{\underset{\sim}{R}}\underset{\sim}{R}^{-1}} + \underset{\sim}{R}\dot{\underset{\sim}{U}}\underset{\sim}{U}^{-1}\underset{\sim}{R}^{-1} + \cancel{\underset{\sim}{R}\dot{\underset{\sim}{R}}^T} + \underset{\sim}{R}\underset{\sim}{U}^{-1}\dot{\underset{\sim}{U}}\underset{\sim}{R}^T\right)$$
$$\Rightarrow \underset{\sim}{D} = \frac{1}{2}\underset{\sim}{R}\left(\dot{\underset{\sim}{U}}\underset{\sim}{U}^{-1} + \underset{\sim}{U}^{-1}\dot{\underset{\sim}{U}}\right)\underset{\sim}{R}^T$$

$$\left(\begin{array}{c}\underset{\sim}{R}\underset{\sim}{R}^T = \underset{\sim}{I} \\ \dot{\underset{\sim}{R}}\underset{\sim}{R}^T + \underset{\sim}{R}\dot{\underset{\sim}{R}}^T = \underset{\sim}{0}\end{array}\right)$$

and
$$\underset{\sim}{W} = \dot{\underset{\sim}{R}}\underset{\sim}{R}^T + \frac{1}{2}\underset{\sim}{R}\left(\dot{\underset{\sim}{U}}\underset{\sim}{U}^{-1} - \underset{\sim}{U}^{-1}\dot{\underset{\sim}{U}}\right)\underset{\sim}{R}^T$$

$$\left(\dot{\underset{\sim}{R}}\underset{\sim}{R}^T = -\underset{\sim}{R}\dot{\underset{\sim}{R}}^T\right)$$

Note: For any skew symmetric tensor $\boldsymbol{W}$:

$$\underset{\sim}{W}\underset{\sim}{u} = \underset{\sim}{\omega} \times \underset{\sim}{u} \Rightarrow \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix}\begin{bmatrix} 0 & W_{12} & W_{13} \\ W_{21} & 0 & W_{23} \\ W_{31} & W_{32} & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \det\begin{bmatrix} e_1 & e_2 & e_3 \\ -W_{23} & -W_{31} & -W_{12} \\ u_1 & u_2 & u_3 \end{bmatrix}$$

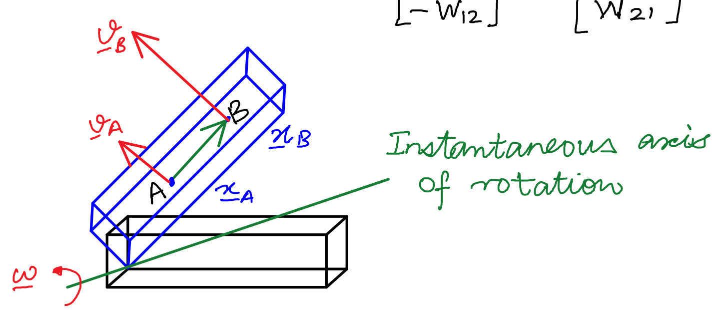
Axial vector of $\underset{\sim}{W}$ (↑ $\underset{\sim}{\omega}$)

$$\Rightarrow \underset{\sim}{\omega} \sim \begin{bmatrix} -W_{23} \\ -W_{31} \\ -W_{12} \end{bmatrix} \sim \begin{bmatrix} W_{32} \\ W_{13} \\ W_{21} \end{bmatrix}$$

Example: Rigid Body Motion:

$$\underset{\sim}{U} = \underset{\sim}{I}\,; \quad \underset{\sim}{F} = \underset{\sim}{R}$$
$$\Rightarrow \underset{\sim}{D} = \underset{\sim}{0}\,; \quad \underset{\sim}{L} = \frac{\partial \underset{\sim}{v}}{\partial \underset{\sim}{x}} = \underset{\sim}{W} = \dot{\underset{\sim}{R}}\underset{\sim}{R}^T$$

$$\Rightarrow d\underset{\sim}{v} = \underset{\sim}{W}\,d\underset{\sim}{x}$$
$$= \underset{\sim}{\omega} \times d\underset{\sim}{x}$$

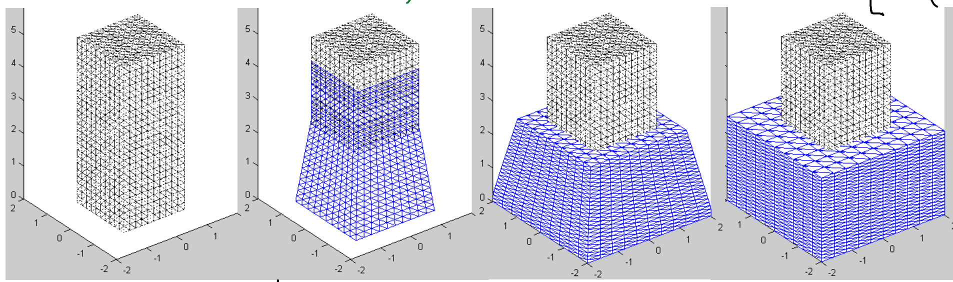


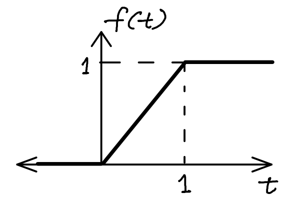
$$\longleftrightarrow \quad \left(\underset{\sim}{v}_B - \underset{\sim}{v}_A\right) = \underset{\sim}{\omega} \times \left(\underset{\sim}{x}_B - \underset{\sim}{x}_A\right)$$

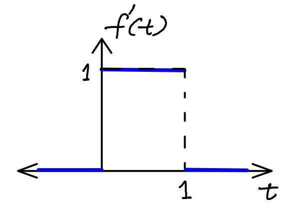
<u>Example</u> Consider a time-dependent map:

$$\underline{\phi}(\underline{z}, t) = \left(1+\alpha f\left(t-\frac{z_3}{V_0}\right)\right) \underline{r} + \left(1+\beta f(t)\right) z_3 \underline{e}_3 \sim \begin{bmatrix} \left(1+\alpha f\left(t-\frac{z_3}{V_0}\right)\right) z_1 \\ \left(1+\alpha f\left(t-\frac{z_3}{V_0}\right)\right) z_2 \\ \left(1+\beta f(t)\right) z_3 \end{bmatrix}$$

$$\underline{r} = (z_1 \underline{e}_1 + z_2 \underline{e}_2)$$



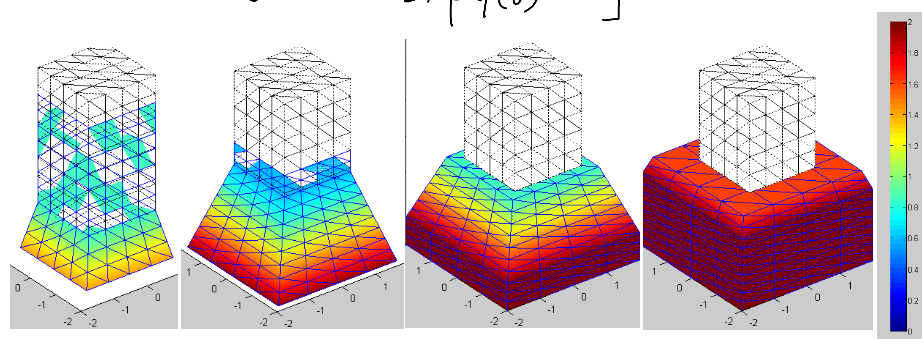




• Deformation Gradient

$$\underset{\sim}{F}(t) \sim \begin{bmatrix} 1+\alpha f\left(t-\frac{z_3}{V_0}\right) & 0 & \alpha z_1 f'\left(t-\frac{z_3}{V_0}\right)\left(\frac{-1}{V_0}\right) \\ 0 & 1+\alpha f\left(t-\frac{z_3}{V_0}\right) & \alpha z_2 f'\left(t-\frac{z_3}{V_0}\right)\left(\frac{-1}{V_0}\right) \\ 0 & 0 & 1+\beta f(t) \end{bmatrix}$$

Plot $\det(\underset{\sim}{F})$:

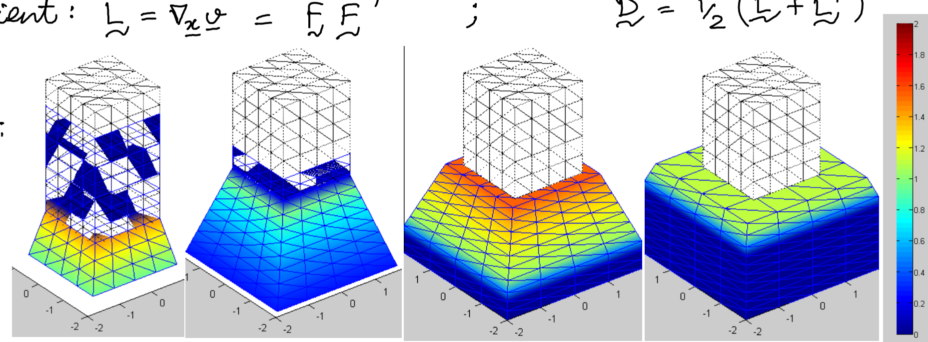


• Rate of Deformation:

$$\dot{\underset{\sim}{F}} \sim \begin{bmatrix} \alpha f'\left(t-\frac{z_3}{V_0}\right) & 0 & 0 \\ 0 & \alpha f'\left(t-\frac{z_3}{V_0}\right) & 0 \\ 0 & 0 & \beta f'(t) \end{bmatrix}$$

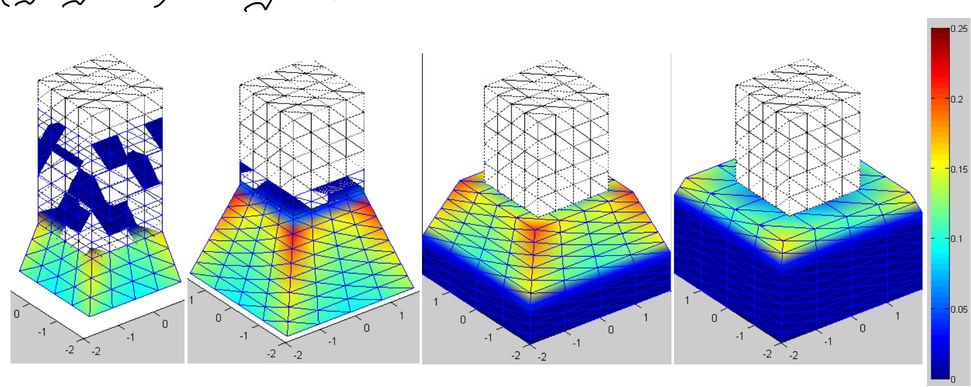
• Spatial velocity gradient: $\underset{\sim}{L} = \nabla_x v = \dot{\underset{\sim}{F}} \underset{\sim}{F}^{-1}$; $\underset{\sim}{D} = \frac{1}{2}(\underset{\sim}{L} + \underset{\sim}{L}^T)$

Plot $tr(\underset{\sim}{D}) = tr(\underset{\sim}{L})$:



• Spin Tensor: $\underset{\sim}{W} = \frac{1}{2}(\underset{\sim}{L} - \underset{\sim}{L}^T)$; $\underset{\sim}{W} = \underset{\sim}{\omega} \times$

Plot of $\|\underset{\sim}{\omega}\|$:



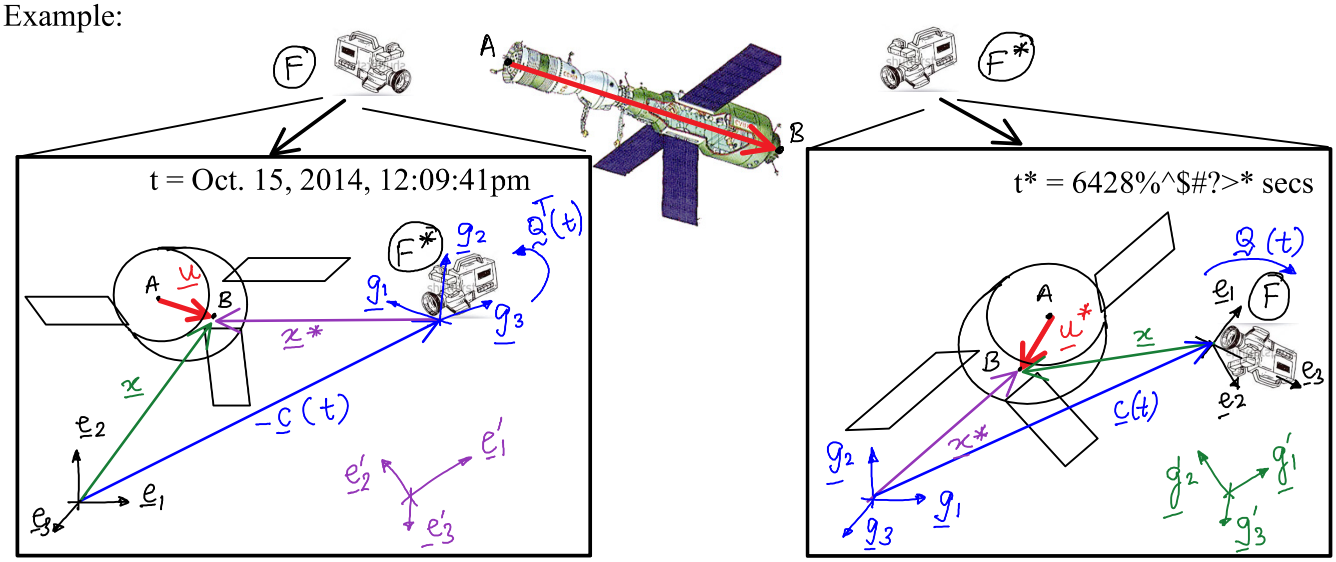
Objectivity / Frame-indifference

It is important that the physical quantities that we use to characterize material behavior and the laws of physics must not change with a change in the frame of reference *i.e.* they must be objective.
While scalar quantities are objective, unfortunately, a lot of vector and tensor quantities (especially those that measure time-rates of changes) are not objective - they are different in different frames of reference.

Frame of reference:

It is a "point of view / way of viewing" the processes occurring in the world / universe.
Think of: Observations (video) from a camera (with full 3D depth perception) and time stamp
(so that you can record the distances, orientations and time-instants precisely).
(It is NOT the same as a choice of a coordinate-system.)

Example:



Frame F		Frame F^*
Time: $t = t^* + \alpha$		$t^* = t - \alpha$
Orientation: $e_i = \underset{\sim}{Q}\, g_i$	Recall $Q_{ij} = g_i \cdot e_j$	$g_i = \underset{\sim}{Q}^T e_i$
Position $x = \underset{\sim}{Q}^T(t)\,(x^* - c(t))$	$\underset{\sim}{Q} = e_j \otimes g_j$	$x^* = c(t) + \underset{\sim}{Q}(t)\, x$
Vectors: $u = \underset{\sim}{Q}^T(t)\, u^*$	$\underset{\sim}{Q}^T = g_j \otimes e_j$	$u^* = \underset{\sim}{Q}(t)\, u$
Tensors: $\underset{\sim}{T} = \underset{\sim}{Q}^T(t)\, \underset{\sim}{T}^*\, \underset{\sim}{Q}(t)$		$\underset{\sim}{T}^* = \underset{\sim}{Q}(t)\, \underset{\sim}{T}\, \underset{\sim}{Q}^T(t)$

Consider velocity and acceleration:

$$v(x,t) = \frac{d x}{dt} = \dot{x}$$

$$a(x,t) = \frac{d v}{dt} = \dot{v}$$

$$v^*(x^*,t) = \frac{d}{dt^*} x^* = \frac{d}{dt} x^* = \dot{x}^*$$

$$v^*(x^*,t) = \dot{c} + \dot{\underset{\sim}{Q}}\, x + \underset{\sim}{Q}\, \underbrace{\dot{x}}_{v} \qquad (\neq \underset{\sim}{Q}\, v)$$

$$a^*(x,t) = \frac{d}{dt^*} v^* = \frac{d v^*}{dt} = \dot{v}^*$$

$$a^*(x,t) = \ddot{c} + \ddot{\underset{\sim}{Q}}\, x + \dot{\underset{\sim}{Q}}\, \dot{x} + \dot{\underset{\sim}{Q}}\, v + \underset{\sim}{Q}\, \underbrace{\ddot{x}}_{a} \qquad (\neq \underset{\sim}{Q}\, a)$$

Thus velocity and acceleration are NOT objective!
Rates of deformation and strain are not objective: $\dot{\underset{\sim}{C}}, \dot{\underset{\sim}{E}}$ But $\underset{\sim}{D} = \frac{1}{2}(\underset{\sim}{L} + \underset{\sim}{L}^T)$ is objective.

Objective Rates: (using Material / Reference frame)

- Co-rotational rate: $\overset{\circ}{u} \equiv \dot{u} - \underset{\sim}{W}\, u$; $\overset{\circ}{\underset{\sim}{T}} \equiv \dot{\underset{\sim}{T}} - \underset{\sim}{W}\,\underset{\sim}{T} + \underset{\sim}{T}\,\underset{\sim}{W}$ (Jaumann)
- Convected rate: $\hat{u} \equiv \dot{u} + \underset{\sim}{L}^T u$; $\hat{\underset{\sim}{T}} \equiv \dot{\underset{\sim}{T}} + \underset{\sim}{L}^T \underset{\sim}{T} + \underset{\sim}{T}\,\underset{\sim}{L}$ (Cotter-Rivlin)
- Oldroyd rate; Truesdell rate; Green-Naghdi rate ...