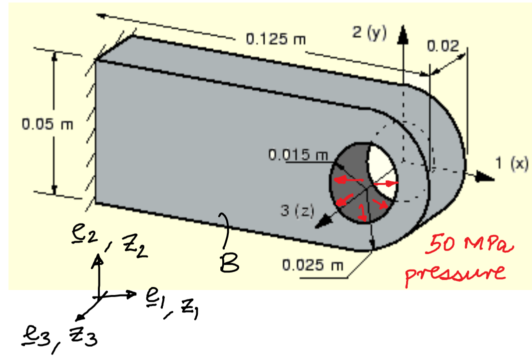
Chapter 4: Material Behavior

Recall governing equations:

GDE
$$\begin{cases} \operatorname{div} \underset{\sim}{\sigma} + \underline{b} = \underline{0} & \text{— EQ} \quad \forall \underline{x} \in \phi(B) \\ \underset{\sim}{\sigma} = \underset{\sim}{\sigma}^T & \forall \underline{x} \in \phi(B) \\ \underset{\sim}{\epsilon} = \tfrac{1}{2}\left(\nabla \underline{u} + \nabla \underline{u}^T\right) & \text{— SD} \quad \forall \underline{x} \in \phi(B) \end{cases}$$

BC
$$\begin{cases} \underline{u} = \underline{u}_D & \forall \underline{x} \in \phi(A_D) \\ \underline{t}_{(n)} = \underset{\sim}{\sigma}\, \underline{n} = \underline{h}_N & \forall \underline{x} \in \phi(A_N) \end{cases}$$



Unknowns : $\underbrace{\underline{u}(\underline{x})}_{3} + \underbrace{\underset{\sim}{\epsilon}(\underline{x})}_{6} + \underbrace{\underset{\sim}{\sigma}(\underline{x})}_{6}$: 15 unknowns $\forall \underline{x} \in \phi(B)$

Equations : EQ SD ? : Need 6 "constitutive" equations

Different materials behave differently when subjected to loads and deformation.
Thus stress-strain relationships depend upon material properties that have to be determined by conducting experiments in a lab (theory alone does not suffice).

General principles for material stress-strain (constitutive) relationships:

- Deterministic
- Local action
- Observer objectivity / Material frame indifference
- Laws of Thermodynamics

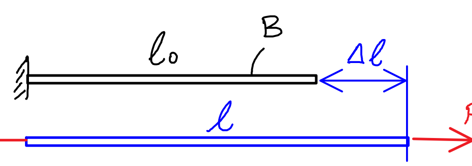
Types of material models:

- Elastic: Hyper-elastic, Hypo-elastic, Visco-elastic
- Inelastic: Plastic, Visco-plastic, Damage models ...

Elasticity

A fundamental tenet of elasticity is that when a loaded material (body) is unloaded, it returns to its original undeformed shape/state.
Another way of saying this is that stress-strain relationship at a point does *not* depend upon how / in what sequence of operations (history) the material point was loaded i.e. path-independence.
It only depends on the current *state* of strain at the point.

Example: 1D Linear Hooke's law (model)

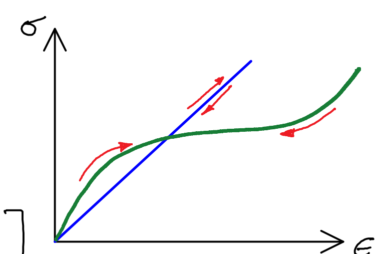


$$F = k\,\Delta l \;;$$

$$\frac{F}{A} = \left(\frac{k\, l_0}{A}\right)\frac{\Delta l}{l_0} \Rightarrow \sigma = C\,\epsilon$$

Young's Mod.

In general, stress-strain relationship may not be linear: $\sigma(\epsilon)$



Note: Work done = Energy stored

$$W_{TOT} = \int_0^{\Delta l} F \, dl = \int_0^{\Delta l} \left(\frac{F}{A}\right)\left(\frac{dl}{l_0}\right)\underbrace{(A\, l_0)}_{V(B)} = \int_{V(B)} \underbrace{\left[\int_0^{\epsilon} \sigma \, d\epsilon\right]}_{\psi(\epsilon)} dV$$

$\psi(\epsilon)$: Strain Energy Density func.

These are Hyper-elastic materials.

Strain energy (density) function

Stores the energy resulting from internal work done at each point of a body /structure.

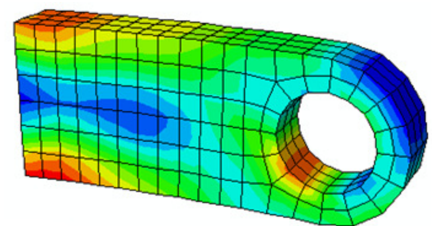
$$\psi(\epsilon) = \int_0^{\epsilon} \sigma(\epsilon)\, d\epsilon \quad \Rightarrow \quad \sigma(\epsilon) = \frac{d\psi}{d\epsilon}$$

If stress-strain relationship is linear (Hooke's model):

$$\psi(\epsilon) = \tfrac{1}{2} C \epsilon^2 \quad \text{and} \quad \sigma = C\epsilon \quad \Rightarrow \quad C = \frac{\partial \sigma}{\partial \epsilon} = \frac{\partial^2 \psi}{\partial \epsilon^2}$$

Generalization to 3D:

stress and strain are tensors.
$(\underset{\sim}{S})$ $(\underset{\sim}{\epsilon})$



Work done at a point

$$\psi(\underset{\sim}{\epsilon}) = \int_{\underset{\sim}{0}}^{\underset{\sim}{\epsilon}} \underset{\sim}{S} : \underbrace{d\underset{\sim}{\epsilon}}_{S_{ij}(d\epsilon)_{ij}} = \int_0^t \underbrace{\underset{\sim}{S} : \dot{\underset{\sim}{\epsilon}}}_{\text{Stress power (Rate of internal work done)}} dt \qquad \frac{d\underset{\sim}{\epsilon}}{dt}$$

Stress power
(Rate of internal work done)

$$\frac{d\psi}{dt}$$

$$\text{Thus} \quad \dot{\psi} = \underset{\sim}{S} : \dot{\underset{\sim}{\epsilon}} \quad \text{and noting} \quad \dot{\psi} = \frac{\partial \psi}{\partial \epsilon_{ij}} : \dot{\epsilon}_{ij}$$

chain Rule

$$\Rightarrow \quad \boxed{\underset{\sim}{S} = \frac{\partial \psi}{\partial \underset{\sim}{\epsilon}}} \quad \Rightarrow \quad S_{ij} = \frac{\partial \psi}{\partial \epsilon_{ij}}$$

If stress-strain relationship is linear:

$$\underset{\sim}{S} = \underset{\approx}{C}\, \underset{\sim}{\epsilon} \quad \Rightarrow \quad S_{ij} = C_{ijkl}\, \epsilon_{kl}$$

4th order tensor (~~81~~ elastic constants)

~~36~~ : Symmetry of $\underset{\sim}{S}$; $\underset{\sim}{\epsilon}$

21 : Symmetry of $\underset{\approx}{C}$

and

$$C_{ijkl} = \frac{\partial S_{ij}}{\partial \epsilon_{kl}} = \frac{\partial^2 \psi}{\partial \epsilon_{ij}\, \partial \epsilon_{kl}} \quad ; \quad \underset{\approx}{C} = C_{ijkl}\left(\underset{-}{e}_i \otimes \underset{-}{e}_j \otimes \underset{-}{e}_k \otimes \underset{-}{e}_l\right)$$

One can show: $\psi(\underset{\sim}{\epsilon}) = \tfrac{1}{2}\, \epsilon_{ij}\, C_{ijkl}\, \epsilon_{kl}$

Aside: 4th order Tensors:

$$\text{If} \quad \underset{\approx}{T} = [\underline{s}\otimes\underline{t}\otimes\underline{u}\otimes\underline{v}] \;;\quad \underset{\sim}{S} = (\underline{a}\otimes\underline{b})$$

$$\text{Then} \quad \underset{\approx}{T}\,\underset{\sim}{S} \equiv [(\underline{s}\otimes\underline{t})\otimes(\underline{u}\otimes\underline{v})](\underline{a}\otimes\underline{b}) = (\underline{u}\cdot\underline{a})(\underline{v}\cdot\underline{b})(\underline{s}\otimes\underline{t})$$

$$\text{Note} \quad [(\underline{e}_i\otimes\underline{e}_j)\otimes(\underline{e}_k\otimes\underline{e}_l)][\underline{e}_m\otimes\underline{e}_n] = \delta_{km}\delta_{ln}(\underline{e}_i\otimes\underline{e}_j)$$

$$\Rightarrow \quad S_{ij}(\underline{e}_i\otimes\underline{e}_j) = C_{ijkl}[(\underline{e}_i\otimes\underline{e}_j)\otimes(\underline{e}_k\otimes\underline{e}_l)]\,\epsilon_{\cancel{mn}\,kl}(\underline{e}_m\otimes\underline{e}_n)$$

$$\delta_{k\cancel{m}} \qquad \delta_{l\cancel{n}}$$

$$\text{Thus} \quad S_{ij} = C_{ijkl}\,\epsilon_{kl}$$

Voight notation

For convenience of visualization (and ease of computer implementation), the relationship between stress and strain is sometimes expressed in a different notation using matrices (Problem 101):

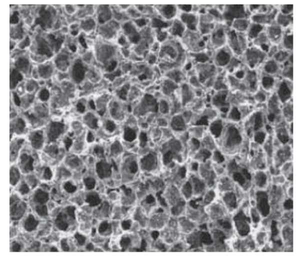
$$\begin{bmatrix} S_{11} \\ S_{22} \\ S_{33} \\ \hline S_{12} \\ S_{23} \\ S_{31} \\ S_{21} \\ S_{32} \\ S_{13} \end{bmatrix} = \left[\begin{array}{ccc|ccc|ccc} C_{1111} & C_{1122} & C_{1133} & C_{1112} & C_{1123} & C_{1131} & C_{1121} & C_{1132} & C_{1113} \\ C_{2211} & C_{2222} & C_{2233} & C_{2212} & C_{2223} & C_{2231} & C_{2221} & C_{2232} & C_{2213} \\ C_{3311} & C_{3322} & C_{3333} & C_{3312} & C_{3323} & C_{3331} & C_{3321} & C_{3332} & C_{3313} \\ \hline C_{1211} & C_{1222} & C_{1233} & C_{1212} & C_{1223} & C_{1231} & C_{1221} & C_{1232} & C_{1213} \\ C_{2311} & C_{2322} & C_{2333} & C_{2312} & C_{2323} & C_{2331} & C_{2321} & C_{2332} & C_{2313} \\ C_{3111} & C_{3122} & C_{3133} & C_{3112} & C_{3123} & C_{3131} & C_{3121} & C_{3132} & C_{3113} \\ \hline C_{2111} & C_{2122} & C_{2133} & C_{2112} & C_{2123} & C_{2131} & C_{2121} & C_{2132} & C_{2113} \\ C_{3211} & C_{3222} & C_{3233} & C_{3212} & C_{3223} & C_{3231} & C_{3221} & C_{3232} & C_{3213} \\ C_{1311} & C_{1322} & C_{1333} & C_{1312} & C_{1323} & C_{1331} & C_{1321} & C_{1332} & C_{1313} \end{array}\right] \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \hline \epsilon_{12} \\ \epsilon_{23} \\ \epsilon_{31} \\ \epsilon_{21} \\ \epsilon_{32} \\ \epsilon_{13} \end{bmatrix}$$

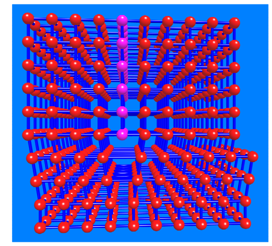
Taking advantage of symmetry as:

$$\begin{bmatrix} S_{11} \\ S_{22} \\ S_{33} \\ \hline S_{12} \\ S_{23} \\ S_{31} \end{bmatrix} = \left[\begin{array}{ccc|ccc} C_{1111} & C_{1122} & C_{1133} & C_{1112} & C_{1123} & C_{1131} \\ C_{2211} & C_{2222} & C_{2233} & C_{2212} & C_{2223} & C_{2231} \\ C_{3311} & C_{3322} & C_{3333} & C_{3312} & C_{3323} & C_{3331} \\ \hline C_{1211} & C_{1222} & C_{1233} & C_{1212} & C_{1223} & C_{1231} \\ C_{2311} & C_{2322} & C_{2333} & C_{2312} & C_{2323} & C_{2331} \\ C_{3111} & C_{3122} & C_{3133} & C_{3112} & C_{3123} & C_{3131} \end{array}\right] \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \hline 2\epsilon_{12} \\ 2\epsilon_{23} \\ 2\epsilon_{31} \end{bmatrix}$$

<u>Isotropy</u>

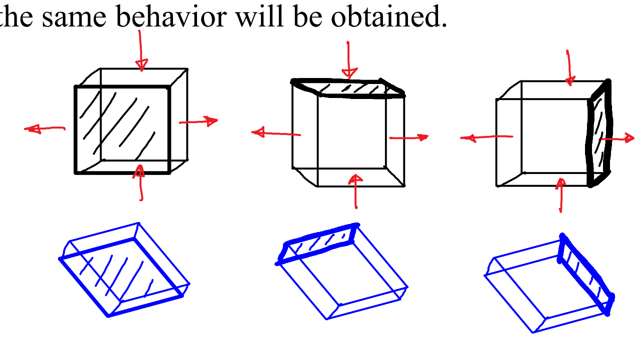
Materials derive their properties from their internal micro-structure i.e. from the make up of their constituent atoms and molecules.
If this micro-structure is uniform and homogenous in all directions, then the material displays same behavior in all directions of loading and is called <u>isotropic</u> e.g. steel , plain concrete...

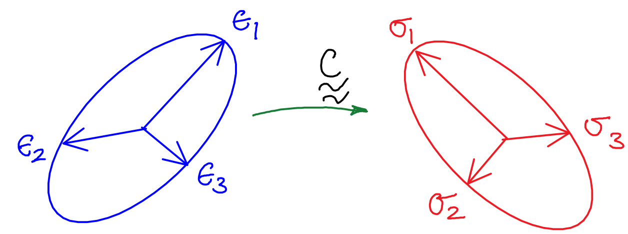




Note: Most materials are <u>not</u> isotropic e.g. wood, reinforced composites, natural materials...

If a piece of isotropic material is subjected to the same state of stress and strain, but in different orientations, the same behavior will be obtained.





One way to define isotropic hyper elastic materials is in terms of the invariants of strain.
An energy density function in terms of invariants is isotropic:

$$\hat{\psi}\Big(\underbrace{tr(\underset{\sim}{\epsilon})}_{f_1(\underset{\sim}{\epsilon})},\ \underbrace{tr(\underset{\sim}{\epsilon}^2)}_{f_2(\underset{\sim}{\epsilon})},\ \underbrace{tr(\underset{\sim}{\epsilon}^3)}_{f_3(\underset{\sim}{\epsilon})}\Big)$$

Thus
$$\underset{\sim}{S} = \frac{\partial \psi}{\partial \underset{\sim}{\epsilon}} = \frac{\partial \psi}{\partial f_1}\cdot\underbrace{\frac{\partial f_1}{\partial \underset{\sim}{\epsilon}}}_{\underset{\sim}{I}} + \frac{\partial \psi}{\partial f_2}\cdot\underbrace{\frac{\partial f_2}{\partial \underset{\sim}{\epsilon}}}_{2\underset{\sim}{\epsilon}} + \frac{\partial \psi}{\partial f_3}\cdot\underbrace{\frac{\partial f_3}{\partial \underset{\sim}{\epsilon}}}_{3\underset{\sim}{\epsilon}^2}$$

$$\frac{\partial f_1}{\partial \underset{\sim}{\epsilon}} = \frac{\partial \epsilon_{kk}}{\partial \epsilon_{ij}}(\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j) = \delta_{i\cancel{k}_j}\,\delta_{j\cancel{k}}\,(\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j) = \underset{\sim}{I}$$

$$\frac{\partial f_2}{\partial \underset{\sim}{\epsilon}} = \frac{\partial \epsilon_{kl}\epsilon_{lk}}{\partial \epsilon_{ij}}(\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j) = \left(\cancel{\delta_{ki}}\,\cancel{\delta_{lj}}\ \epsilon_{\cancel{l}_j\cancel{k}_i} + \epsilon_{\cancel{k}_j\cancel{l}_i}\,\cancel{\delta_{li}}\,\cancel{\delta_{kj}}\right)\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j = 2\underset{\sim}{\epsilon}$$

$$\frac{\partial f_3}{\partial \underset{\sim}{\epsilon}} = \frac{\partial \epsilon_{kl}\epsilon_{lm}\epsilon_{mk}}{\partial \epsilon_{ij}}(\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j)$$

$$= \left[\cancel{\delta_{ki}}\,\cancel{\delta_{lj}}\ \epsilon_{\cancel{l}_jm}\,\epsilon_{m\cancel{k}_i} + \cancel{\delta_{li}}\,\cancel{\delta_{mj}}\ \epsilon_{k\cancel{l}_i}\,\epsilon_{\cancel{m}_jk} + \cancel{\delta_{mi}}\,\cancel{\delta_{kj}}\ \epsilon_{\cancel{k}_jl}\,\epsilon_{l\cancel{m}_i}\right](\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j) = 3\underset{\sim}{\epsilon}^2$$

Thus
$$\underset{\sim}{S} = \boxed{\frac{\partial \psi}{\partial f_1}}\cdot\underset{\sim}{I} + \boxed{\frac{\partial \psi}{\partial f_2}\cdot 2}\,\underset{\sim}{\epsilon} + \boxed{\frac{\partial \psi}{\partial f_3}\cdot 3}\,\underset{\sim}{\epsilon}^2$$

scalar material properties.

Linear Isotropic Hyper elastic material:

Strain energy density function must be quadratic: $\psi(\boldsymbol{\epsilon}) = \alpha_1 [tr(\boldsymbol{\epsilon})]^2 + \alpha_2\, tr(\boldsymbol{\epsilon}^2)$

A particular form of this is: $\psi(\boldsymbol{\epsilon}) = \frac{\lambda}{2}[tr(\boldsymbol{\epsilon})]^2 + \mu\, tr(\boldsymbol{\epsilon}^2)$

$$\Rightarrow \quad \boldsymbol{S} = \frac{\partial \psi}{\partial f_1}\cdot \boldsymbol{I} + \frac{\partial \psi}{\partial f_2}\cdot 2\boldsymbol{\epsilon} \quad \Rightarrow \quad \boxed{\boldsymbol{S} = \lambda\, tr(\boldsymbol{\epsilon})\boldsymbol{I} + 2\mu\,\boldsymbol{\epsilon}}$$

Lamé parameters: λ, μ
(2 constants)

This is the 3D version of the Hooke' (law) model.

Note: $$C_{ijkl} = \frac{\partial S_{ij}}{\partial \epsilon_{kl}} = \frac{\partial(\lambda\, \epsilon_{mm}\delta_{ij} + 2\mu\, \epsilon_{ij})}{\partial \epsilon_{kl}}$$

$$= \lambda\, \delta_{\cancel{m}k}\, \delta_{\overset{k}{\cancel{m}}l}\, \delta_{ij} + 2\mu\, \delta_{ik}\delta_{jl}$$

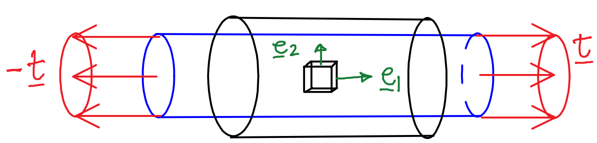
Alternatively $$\boxed{C_{ijkl} = \lambda\, \delta_{ij}\delta_{kl} + \mu\,[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}]}$$

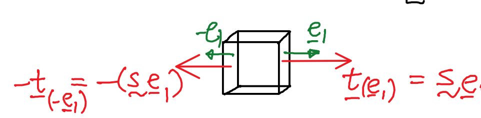
Using this we can express: $S_{ij} = C_{ijkl}\,\epsilon_{kl}$

Another useful form: (given stress, find strain): $$\boxed{\boldsymbol{\epsilon} = -\frac{\lambda}{2\mu(3\lambda+2\mu)}\, tr(\boldsymbol{S})\boldsymbol{I} + \frac{1}{2\mu}\boldsymbol{S}}$$

Finding Material Constants Experimentally: 1: Uniaxial tension

Recall Uniaxial Tension: $\boldsymbol{S} = \sigma_1(\boldsymbol{e}_1 \otimes \boldsymbol{e}_1) \sim \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$; $\boldsymbol{\epsilon} \sim \begin{bmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{bmatrix}$





Solving for strains using the Hooke's model:

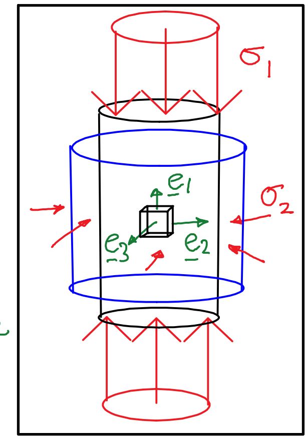
$$\boldsymbol{\epsilon} = \frac{-\lambda}{2\mu(3\lambda+2\mu)}\sigma_1 \boldsymbol{I} + \frac{1}{2\mu}\boldsymbol{S} \quad \Rightarrow \quad \epsilon_{11} = \epsilon_1 = \frac{(\lambda+\mu)}{\mu(3\lambda+2\mu)}\sigma_1$$

$$\Rightarrow \quad \text{Young's Modulus: } \boxed{C = \frac{\sigma_1}{\epsilon_1} = \frac{\mu(3\lambda+2\mu)}{(\lambda+\mu)}}$$

Note: $\epsilon_1 = \frac{\partial u_1}{\partial z_1} \approx \frac{\Delta l}{l_0}$

Similarly $$\epsilon_{22} = \epsilon_{33} = \frac{-\lambda}{2\mu(3\lambda+2\mu)}\sigma = \frac{-\lambda}{2(\lambda+\mu)}\epsilon_{11}$$

$$\Rightarrow \quad \text{Poisson's Ratio: } \boxed{\nu = \frac{-\epsilon_{22}}{\epsilon_{11}} = \frac{-\epsilon_{33}}{\epsilon_{11}} = \frac{\lambda}{2(\lambda+\mu)}}$$



pressure chamber $(-\sigma_2)$

For a triaxial test:

$$\underset{\sim}{S} = \sigma_1 (\underset{\sim}{e}_1 \otimes \underset{\sim}{e}_1) + \sigma_2 \left[(\underset{\sim}{e}_2 \otimes \underset{\sim}{e}_2) + \underset{\sim}{e}_3 \otimes \underset{\sim}{e}_3 \right]$$

$$\underset{\sim}{\epsilon} = \epsilon_1 (\underset{\sim}{e}_1 \otimes \underset{\sim}{e}_1) + \epsilon_2 \left[(\underset{\sim}{e}_2 \otimes \underset{\sim}{e}_2) + (\underset{\sim}{e}_3 \otimes \underset{\sim}{e}_3) \right]$$

Note:

- $tr(\underset{\sim}{S}) = (\sigma_1 + 2\sigma_2) = 3\sigma$ $\quad \left\{ \sigma\text{: Equivalent pressure in sample (suction)} \right.$
- $tr(\underset{\sim}{\epsilon}) = (\epsilon_1 + 2\epsilon_2) = e$ $\quad \left\{ e\text{: Dilatation} \right.$
- Change in volume: $\dfrac{v}{V} = \det(\underset{\sim}{F}) = \sqrt{\det(\underset{\sim}{C})} = \sqrt{\det(\underset{\sim}{I} + 2\underset{\sim}{E})}$

  One can show that $\dfrac{v}{V} = \det(\underset{\sim}{F}) \approx 1 + tr(\underset{\sim}{E}) \approx 1 + tr(\underset{\sim}{\epsilon})$

$\Rightarrow$ Dilatation: $e = \dfrac{\Delta V}{V} = \dfrac{v - V}{V} = \det(\underset{\sim}{F}) - 1 \approx tr(\underset{\sim}{\epsilon})$

From the Hooke's model: $tr(\underset{\sim}{S}) = \lambda\, tr(\underset{\sim}{\epsilon})\, tr(\underset{\sim}{I}) + 2\mu\, tr(\underset{\sim}{\epsilon})$

$$\Rightarrow \underbrace{tr(\underset{\sim}{S})}_{3\sigma} = \underbrace{(3\lambda + 2\mu)}_{3K} \underbrace{tr(\underset{\sim}{\epsilon})}_{e}$$

$$\Rightarrow \boxed{\sigma = K e}$$

Bulk modulus $\left\{ K = \dfrac{\sigma}{e} = \left(\lambda + \dfrac{2}{3}\mu\right) \right.$

Define Deviatoric Stress: $\underset{\sim}{S}' = \underset{\sim}{S} - \frac{1}{3} tr(\underset{\sim}{S}) \underset{\sim}{I} = \underset{\sim}{S} - \sigma \underset{\sim}{I}$ $\quad \left\{ \text{Note: } tr(\underset{\sim}{S}') = 0 \right.$

Again from Hooke's model:

$$\underset{\sim}{S}' = \underbrace{\lambda\, tr(\underset{\sim}{\epsilon}) \underset{\sim}{I} + 2\mu (\underset{\sim}{\epsilon})}_{\underset{\sim}{S}} - \sigma \underset{\sim}{I} = 2\mu \underbrace{\left(\underset{\sim}{\epsilon} - \frac{1}{3} tr(\underset{\sim}{\epsilon}) \underset{\sim}{I} \right)}_{\underset{\sim}{\epsilon}'\text{: Deviatoric Strain.}}$$

Thus $\boxed{\underset{\sim}{S}' = 2\mu\, \underset{\sim}{\epsilon}'}$

In terms of K, μ: $\boxed{\underset{\sim}{S} = \underset{\sim}{S}' - \overbrace{\tfrac{1}{3} tr(\underset{\sim}{S})}^{\sigma} \underset{\sim}{I} = K\, \overbrace{tr(\underset{\sim}{\epsilon})}^{e}\, \underset{\sim}{I} + 2\mu\, \underset{\sim}{\epsilon}'}$

Thus by measuring σ_1, σ_2, e, and ϵ_1 we can find

$$K = \frac{\sigma}{e} = \frac{(\sigma_1 + 2\sigma_2)}{3e} \quad ; \quad \epsilon_2 = \frac{e - \epsilon_1}{2}$$

Noting that $\underset{\sim}{S}' = \frac{1}{3}(\sigma_1 - \sigma_2)\left[3(\underset{\sim}{e}_1 \otimes \underset{\sim}{e}_1) - \underset{\sim}{I}\right]$

and $\underset{\sim}{\epsilon}' = \frac{1}{3}(\epsilon_1 - \epsilon_2)\left[3(\underset{\sim}{e}_1 \otimes \underset{\sim}{e}_1) - \underset{\sim}{I}\right]$ $\quad \Bigg\} \Rightarrow 2\mu = \dfrac{\sigma_1 - \sigma_2}{\epsilon_1 - \epsilon_2}$

Useful relationships between material moduli (Problem 98)

All the material moduli are related and can be derived in terms of one another.

For a linear hyper-elastic isotropic material one needs any two of these material constants to describe the relationship between in stress and strain.

$$\lambda = \frac{2\mu\nu}{1-2\nu} = \frac{\mu(C-2\mu)}{3\mu-C} = \frac{C\nu}{(1+\nu)(1-2\nu)} = \frac{3K\nu}{1+\nu}$$

$$K = \boxed{\lambda + \tfrac{2}{3}\mu} = \frac{\mu C}{3(3\mu-C)} = \frac{\lambda(1+\nu)}{3\nu} = \boxed{\frac{C}{3(1-2\nu)}}$$

$$C = 2\mu(1+\nu) = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu} = \frac{\lambda(1+\nu)(1-2\nu)}{\nu} = \frac{9K\mu}{3K+\mu}$$

$$\mu = \boxed{\frac{C}{2(1+\nu)}} = \tfrac{3}{2}(K-\lambda) = \frac{3K(1-2\nu)}{2(1+\nu)} = \frac{\lambda(1-2\nu)}{2\nu}$$

$$\nu = \frac{\lambda}{2(\lambda+\mu)} = \frac{C}{2\mu} - 1 = \frac{3K-2\mu}{2(3K-\mu)} = \frac{3K-C}{6K}$$

Note:

- Under uniaxial tension $\sigma_1 = C\,\epsilon_1$

  If $\sigma_1 > 0 \Rightarrow \epsilon_1 > 0 \Rightarrow C > 0$

- Under hydrostatic pressure state of stress $\underset{\sim}{S} = -p\underset{\sim}{I}$

  $\Rightarrow \frac{1}{3}tr(\underset{\sim}{S}) = -P = \sigma$

  Dilatation: $e < 0 \Rightarrow \frac{\sigma}{K} < 0 \Rightarrow \frac{-P}{K} < 0 \Rightarrow K > 0$

  For Bulk modulus $K > 0 \Rightarrow 1-2\nu > 0 \Rightarrow \nu < 1/2$

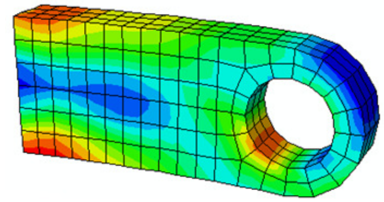
  Further since the Lamé parameter λ can be negative,

  $K = \lambda + \frac{2}{3}\mu \Rightarrow \mu > 0 \Rightarrow 1+\nu > 0 \Rightarrow \nu > -1$

  Thus limits on Poisson's ratio: $\boxed{-1 < \nu < 1/2}$

- Lamé parameter μ is also referred to as "G" Shear Modulus

Hyper-elasticity for finite (large) strains:



Let's revisit the Strain Energy density function:

In spatial (deformed) configuration:

External Work done (over incremental displacement $d\underline{u}$:

$$W_{TOT} = \int_{\phi(A)} d\underline{u}(\underline{x}) \cdot \underline{t}_{(\underline{n})} \, da + \int_{\phi(B)} d\underline{u}(\underline{x}) \cdot \underline{b} \, dv$$

Note: $d\underline{u} \cdot \underline{t}_{(\underline{n})} = d\underline{u} \cdot \boldsymbol{\sigma} \underline{n} = (\boldsymbol{\sigma} \, d\underline{u}) \cdot \underline{n}$

Using Divergence Theorem: $\displaystyle\int_{\phi(A)} (\boldsymbol{\sigma} \, d\underline{u}) \cdot \underline{n} \, da = \int_{\phi(B)} \operatorname{div}(\boldsymbol{\sigma} \, d\underline{u}) \, dv$

Also note: $\operatorname{div}(\boldsymbol{\sigma} \, d\underline{u}) = \operatorname{div}(\boldsymbol{\sigma}) \cdot d\underline{u} + \boldsymbol{\sigma} : \nabla_{\underline{x}}(d\underline{u})$

Thus $$W_{TOT} = \int_{\phi(B)} d\underline{u} \cdot \cancel{(\operatorname{div} \boldsymbol{\sigma} + \underline{b})} \, dv + \int_{\phi(B)} \boldsymbol{\sigma} : \nabla_{\underline{x}} \overbrace{\frac{d\underline{u}}{dt}}^{v} \, dt \, dv$$

Over the entire deformation process:

$$W_{TOT} = \int_{\phi(B)} \left[\int_0^t \boldsymbol{\sigma} : \mathbf{L} \, dt \right] dv = \int_B \left[\int_0^t \boldsymbol{\sigma} : \mathbf{L} \, dt \right] \overbrace{\det(\mathbf{F})}^{J} \, dV = \int_0^t \left[\int_B J \boldsymbol{\sigma} : \overbrace{\mathbf{L}}^{\dot{\mathbf{F}} \mathbf{F}^{-1}} \, dV \right] dt$$

It can be shown that $\displaystyle W_{TOT} = \int_0^t \left[\int_B \underbrace{(J \boldsymbol{\sigma} \mathbf{F}^{-T})}_{\mathbf{P}} : \dot{\mathbf{F}} \, dV \right] dt$

and finally:

$$\boxed{W_{TOT} = \int_{\phi(B)} \left[\int_0^t \boldsymbol{\sigma} : \mathbf{L} \, dv \right] dt = \int_0^t \left[\int_B \mathbf{P} : \dot{\mathbf{F}} \, dV \right] dt = \int_0^t \left[\int_B \mathbf{S} : \dot{\mathbf{E}} \, dV \right] dt}$$

Thus

Cauchy Stress $\boldsymbol{\sigma}$ is work (rate) conjugate to velocity gradient $\mathbf{L}$

1st Piola Kirchhoff stress $\mathbf{P}$ " to rate of deformation gradient $\dot{\mathbf{F}}$

2nd Piola Kirchhoff stress $\mathbf{S}$ " to rate of Green-Lagrange strain $\dot{\mathbf{E}}$

Thus

$$W_{TOT} = \int_0^t \left[\int_{\phi(B)} \dot{\psi}(\nabla_{\underline{x}}(d\underline{u})) \, dv \right] dt = \int_B \left[\int_0^t \underbrace{\dot{\hat{\psi}}(\mathbf{F})}_{\Downarrow} \, dV \right] dt = \int_B \left[\int_0^t \underbrace{\dot{\bar{\Psi}}(\mathbf{E})}_{\Downarrow} \, dV \right] dt$$

$$\mathbf{P} = \frac{\partial \hat{\psi}(\mathbf{F})}{\partial \mathbf{F}} \qquad \mathbf{S} = \frac{\partial \bar{\Psi}(\mathbf{E})}{\partial \mathbf{E}}$$

Examples of Isotropic Hyper-elastic Models:

Note that $\bar{\bar{\Psi}}(\underset{\sim}{E}) = \bar{\Psi}(\underset{\sim}{C})$ such that $\underset{\sim}{S} = \dfrac{\partial \bar{\bar{\Psi}}}{\partial \underset{\sim}{E}} = 2\dfrac{\partial \bar{\Psi}}{\partial \underset{\sim}{C}}$

Once again for isotropic hyper-elastic models:

$$\underset{\sim}{S} = 2\frac{\partial \bar{\Psi}}{\partial \underset{\sim}{C}} = 2\left[\frac{\partial \bar{\Psi}}{\partial f_1}\cdot\underbrace{\frac{\partial f_1}{\partial \underset{\sim}{C}}}_{\underset{\sim}{I}} + \frac{\partial \bar{\Psi}}{\partial f_2}\cdot\underbrace{\frac{\partial f_2}{\partial \underset{\sim}{C}}}_{2\underset{\sim}{C}} + \frac{\partial \Psi}{\partial f_3}\cdot\underbrace{\frac{\partial f_3}{\partial \underset{\sim}{C}}}_{3\underset{\sim}{C}^2}\right]$$

- Mooney-Rivlin model: $\bar{\Psi}(\underset{\sim}{C}) = a(I_c - 3) + b(II_c - 3)$

- St. Venant Kirchhoff model: $\bar{\bar{\Psi}}(\underset{\sim}{E}) = \dfrac{\lambda}{2}\,tr(\underset{\sim}{E})^2 + \mu\, tr(\underset{\sim}{E}^2)$

  (or improved): $\bar{\bar{\Psi}}(\underset{\sim}{E}) = \dfrac{\lambda}{2}(\ln J)^2 + \mu\, tr(\underset{\sim}{E}^2)$

- Neo-Hookean model (Compressible):

$$\bar{\Psi}(\underset{\sim}{C}) = \frac{\mu}{2}(I_c - 3) - \mu \ln J + \frac{\mu}{2}(\ln J)^2$$

Note: Perfectly incompressible (and nearly incompressible) materials require special treatment to ensure det ($\boldsymbol{F}$) = J =1