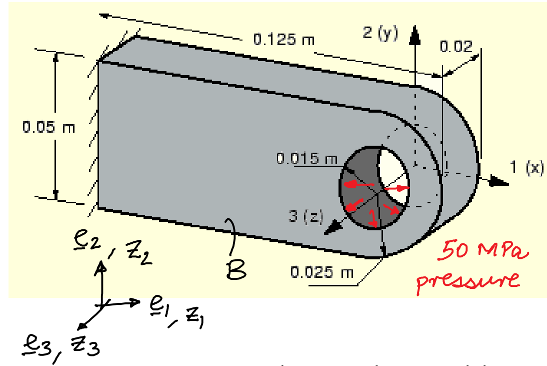
Chapter 5: Boundary value problems in solid mechanics

Recall governing equations:

GDE
$$\begin{cases} \operatorname{div} \underset{\sim}{S} + \underline{b} = \underline{0} \quad \text{— (EQ)} & \forall \underline{x} \in \phi(B) \\ \underset{\sim}{S} = \underset{\sim}{S}^T & \forall \underline{x} \in \phi(B) \\ \underset{\sim}{\varepsilon} = \tfrac{1}{2}\left(\underset{\sim}{\nabla} \underline{u} + \underset{\sim}{\nabla} \underline{u}^T\right) \quad \text{— (SD)} & \forall \underline{x} \in \phi(B) \\ \underset{\sim}{S} = \lambda \, tr(\underset{\sim}{\varepsilon}) \underset{\sim}{I} + 2\mu \underset{\sim}{\varepsilon} \quad \text{— (SS)} & \forall \underline{x} \in \phi(B) \end{cases}$$

BC
$$\begin{cases} \underline{u} = \underline{u}_D & \forall \underline{x} \in \phi(A_D) \\ \underline{t}_{(n)} = \underset{\sim}{S} \, \underline{n} = \underline{h}_N & \forall \underline{x} \in \phi(A_N) \end{cases}$$



Boundary Value Problem

Example: Pure bending of a prismatic cantilever beam:
(pages 250-255, Timoshenko & Goodier)

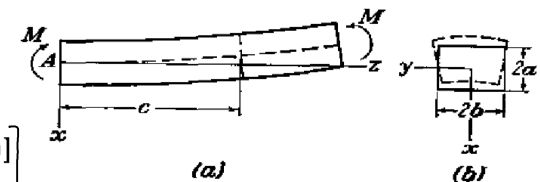


FIG. 141.

Map: $\underline{x} = \underline{z} + \underline{u}(\underline{z})$

$$\underline{x} \sim \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} \qquad \underline{z} \sim \begin{Bmatrix} z_1 \rightarrow x \\ z_2 \rightarrow y \\ z_3 \rightarrow z \end{Bmatrix} \qquad \underline{u} \sim \begin{Bmatrix} u = -\dfrac{1}{2R}[z^2 + \nu(x^2 - y^2)] \\ v = -\dfrac{\nu x y}{R} \\ w = \dfrac{xz}{R} \end{Bmatrix}$$

Note: For a cross section at $z = c$:

$$x_3 = c + w = c + \frac{cx}{R}$$

These kinds of maps are solved for using techniques for PDEs. But work only for simple geometries and loadings.

Note: For the lateral surfaces of the beam:

$$x_2 = \pm b + v = \pm b\left(1 - \frac{\nu x}{R}\right)$$

$$x_1 = \pm a + u = \pm a - \frac{1}{2R}[c^2 + \nu(a^2 - y^2)]$$

Strains: $\underset{\sim}{\varepsilon} = \tfrac{1}{2}\left(\underset{\sim}{\nabla} \underline{u} + \underset{\sim}{\nabla} \underline{u}^T\right)$

$$\underset{\sim}{\varepsilon} \sim \frac{1}{2}\left\{ \begin{bmatrix} -\frac{\nu x}{R} & \cancel{\frac{\nu y}{R}} & \cancel{-\frac{z}{R}} \\ \cancel{-\frac{\nu y}{R}} & -\frac{\nu x}{R} & 0 \\ \cancel{z/R} & 0 & \frac{x}{R} \end{bmatrix} + \begin{bmatrix} -\frac{\nu x}{R} & \cancel{-\frac{\nu y}{R}} & \cancel{z/R} \\ \cancel{\frac{\nu y}{R}} & -\frac{\nu x}{R} & 0 \\ \cancel{-z/R} & 0 & x/R \end{bmatrix} \right\} \sim \begin{bmatrix} -\frac{\nu x}{R} & 0 & 0 \\ 0 & -\frac{\nu x}{R} & 0 \\ 0 & 0 & \frac{x}{R} \end{bmatrix}$$

Stress:

$$\underset{\sim}{S} = \frac{C\nu}{(1+\nu)\cancel{(1-2\nu)}} \underbrace{tr(\underset{\sim}{\varepsilon})}_{\frac{x}{R}\cancel{(1-2\nu)}} \underset{\sim}{I} + \frac{C}{(1+\nu)} \underset{\sim}{\varepsilon} \sim \frac{C}{\cancel{(1+\nu)}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \cancel{(1+\nu)}\frac{x}{R} \end{bmatrix}$$

$$\Rightarrow S_{33} = \boxed{\frac{Cx}{R}}$$

$\left\{ \text{Recall : } \dfrac{\sigma}{y} = \dfrac{E}{R} = \dfrac{M}{I} \right\}$

assumed "0"

Note: $\operatorname{div} \underset{\sim}{S} + \cancel{\underline{b}} = 0 \quad + \text{ Check BCs.}$

Read Example 27 from the textbook.
(See. 101 from Timoshenko & Goodier)

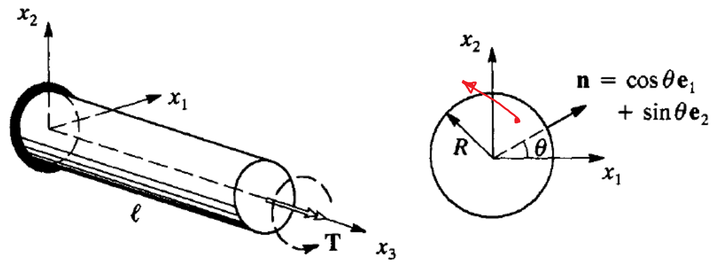


Figure 63 Pure torsion of a circular shaft

Map: $\underset{\sim}{x} = \underset{\sim}{z} + \underset{\sim}{u}(\underset{\sim}{z})$

$$\underset{\sim}{x} = \underset{\sim}{z} \underbrace{- \beta z_2 z_3 \underset{\sim}{e_1} + \beta z_1 z_3 \underset{\sim}{e_2}}_{\underset{\sim}{u}(\underset{\sim}{z})}$$

Strains:

$$\underset{\sim}{\epsilon} \sim \frac{1}{2}\beta \begin{bmatrix} 0 & \cancel{-z_3} & -z_2 \\ \cancel{z_3} & 0 & z_1 \\ -z_2 & z_1 & 0 \end{bmatrix}$$

$$\Rightarrow \text{Stress}: \quad \underset{\sim}{S} = \lambda\, \cancel{tr(\underset{\sim}{\epsilon})}\, \underset{\sim}{I} + 2\mu\, \underset{\sim}{\epsilon} = \mu\beta \begin{bmatrix} 0 & 0 & -z_2 \\ 0 & 0 & z_1 \\ -z_2 & z_1 & 0 \end{bmatrix}$$

0 (assumed)

Verify $\quad div\, \underset{\sim}{S} + \cancel{\underline{b}} = \underline{0} \quad$ + Check BCs.

These systems of PDEs, cannot be solved analytically, in general.
We use numerical methods and use simplifying assumptions.

Simplifying Assumptions:

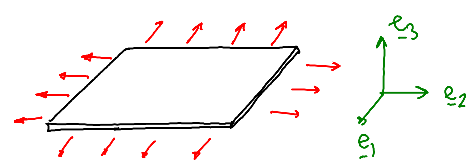
- Axisymmetric problems
- Plane Stress / Plane Strain
- Beam Theory
- Plate / Shell Theory

} Structural Mechanics

Example: 2D Plane Problems

- Plane Stress

$$\boxed{\sigma_{33} = 0 \;;\quad \sigma_{13} = \sigma_{31} = 0 \;;\quad \sigma_{23} = \sigma_{32} = 0}\;;$$



Stress-strain relationship:

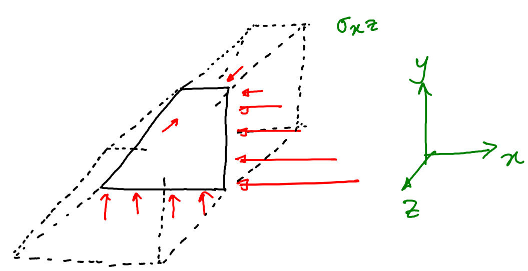
$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ 2\epsilon_{xy} \end{Bmatrix}$$

ie $\boxed{\underline{\sigma} = \underset{\sim}{D}_{P\sigma}\, \underline{\epsilon}}$

and $\epsilon_{zz} = \frac{-\nu}{E}(\sigma_{xx} + \sigma_{yy})$

- Plane Strain

$$\boxed{\epsilon_{33} = 0 \;;\quad \epsilon_{13} = \epsilon_{31} = 0 \;;\quad \epsilon_{23} = \epsilon_{23} = 0}$$



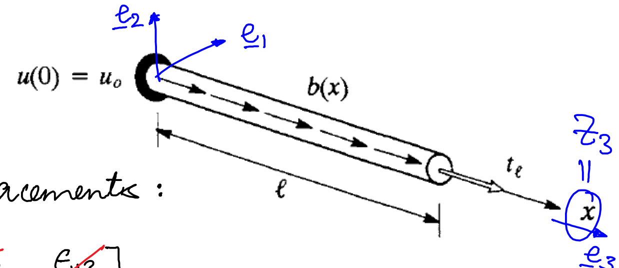
Stress-strain relationship:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & 0 \\ \nu & (1-\nu) & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ 2\epsilon_{xy} \end{Bmatrix}$$

ie $\boxed{\underline{\sigma} = \underset{\sim}{D}_{P\epsilon}\, \underline{\epsilon}}$

One-dimensional (1D) little BVP

To understand how to obtain numerical solutions to complicated 2D/3D problems in general, let's first study some 1-D problems where we can usually obtain exact solutions.



Assuming small strains and displacements:

$$\underset{\sim}{u} \sim \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad u_3 \rightarrow u(x) \qquad \underset{\sim}{\epsilon} \sim \begin{bmatrix} \epsilon_{11} & \cancel{\epsilon_{12}} & \cancel{\epsilon_{13}} \\ \cancel{\epsilon_{21}} & \epsilon_{22} & \cancel{\epsilon_{23}} \\ \cancel{\epsilon_{31}} & \cancel{\epsilon_{32}} & \epsilon_{33} \end{bmatrix} \quad \epsilon_{33} \rightarrow \frac{\partial u_3}{\partial z_3} = u'(x) = \frac{du}{dx}$$

$$\Rightarrow \underset{\sim}{S} = \frac{c\,\nu}{(1+\nu)\cancel{(1-2\nu)}} \underbrace{tr(\underset{\sim}{\epsilon})}_{\cancel{(1-2\nu)}\,\epsilon_{33}} \underset{\sim}{I} + \frac{c}{(1+\nu)} \underset{\sim}{\epsilon} \Rightarrow \frac{c}{\cancel{(1+\nu)}} \begin{bmatrix} \nu\cancel{\epsilon_{33}} + \cancel{\epsilon_{11}} & 0 & 0 \\ 0 & \nu\cancel{\epsilon_{33}} + \cancel{\epsilon_{22}} & 0 \\ & \cancel{(1+\nu)}\,\epsilon_{33} = & \boxed{\nu\epsilon_{33} + \epsilon_{33}} \end{bmatrix} \Rightarrow S_{33} = c\,\epsilon_{33}$$

$$\sigma = c\,\epsilon$$

Thus Governing equations:

GDE
$$\left\{ \begin{array}{ll} \operatorname{div} \underset{\sim}{S} + \underset{\sim}{b} = \underset{\sim}{0} & \Rightarrow \\ \underset{\sim}{\epsilon} = \frac{1}{2}\left(\underset{\sim}{\nabla} \underset{\sim}{u} + \underset{\sim}{\nabla} \underset{\sim}{u}^T\right) & \Rightarrow \\ \underset{\sim}{S} = \lambda\, tr(\underset{\sim}{\epsilon}) \underset{\sim}{I} + 2\mu \underset{\sim}{\epsilon} & \Rightarrow \end{array} \right.$$

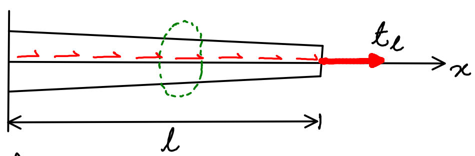
BCs.
$$\left\{ \begin{array}{ll} \underset{\sim}{t}_{(n)} = \underset{\sim}{S} \underset{\sim}{n} = \underset{\sim}{h}_N & \Rightarrow \\ \underset{\sim}{u} = \underset{\sim}{u}_D & \Rightarrow \end{array} \right.$$

$$\boxed{\begin{array}{ll} \frac{d}{dx}\sigma + b(x) = 0 \Rightarrow \sigma' + b = 0 & \\ \epsilon = u' = du/dx & \\ \sigma = c\,\epsilon & \\ t_\ell = \sigma(\ell)(+1) & \\ u(0) = u_o & \end{array}} \quad \begin{array}{l} \forall\, x \in (0,\ell) \\ \forall\, x \in (0,\ell) \\ \forall\, x \in (0,\ell) \\ \text{at } x = \ell \\ \text{at } x = 0 \end{array}$$

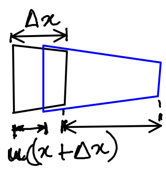
Substituting for $\epsilon(x)$ and $\sigma(x)$:

$$\boxed{\begin{array}{ll} (cu')' + b = 0 & \forall\, x \in (0,\ell) \\ c\,u'(\ell)(+1) = t_\ell & \text{at } x = \ell \\ u(0) = u_o & \text{at } x = 0 \end{array}}$$

Alternative derivation:



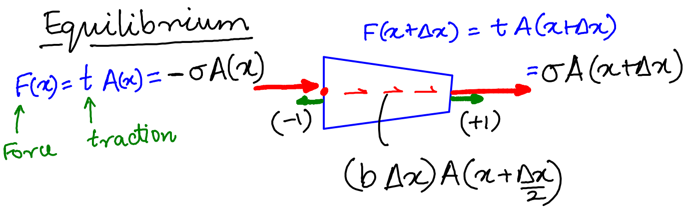
Kinematics:



Strain (1D)

$$\epsilon(x) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x) - u(x)}{\Delta x} \Rightarrow \boxed{\epsilon(x) = \frac{du}{dx} = u'}$$

Equilibrium



Material: $\sigma = c\,\epsilon$

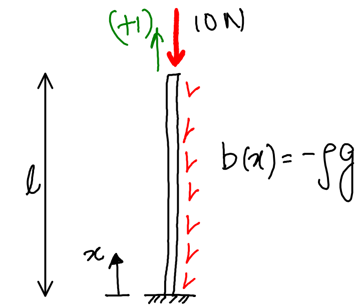
$$\sum F_x = 0 \Rightarrow \sigma A(x+\Delta x) - \sigma A(x) + bA\,\Delta x = 0$$

$$\Rightarrow \lim_{\Delta x \to 0} \frac{\sigma A(x+\Delta x) - \sigma A(x)}{\Delta x} + bA = 0 \Rightarrow \frac{d}{dx}(\sigma A) + bA = 0$$

If $A(x)$ = constant $\Rightarrow$ $\boxed{\sigma' + b = 0}$

Example. Column under self weight:

- $A(x) = \text{constant} = 1$ (say)
- $\sigma(x) = C\,\epsilon(x)$: constant "C"
- Density : ρ
- BCs : $u(0) = 0$ on Γ_D $(x=0)$
  $\sigma(l)(+1) = t_l = -10$ on Γ_L $(x = l)$



Find $\sigma(x)$; $u(x)$.

For stresses:

$$\sigma' + b = 0$$

$$\Rightarrow \frac{d\sigma}{dx} = -b(x) = +\rho g$$

$$\Rightarrow \sigma(x) = \rho g x + c_1$$

$$t_L = \sigma(l) = \rho g l + c_1$$

BC:

$$\sigma(l)(+1) = t_L = -10$$

$$\Rightarrow c_1 = t_L - \rho g l$$

$$\Rightarrow \boxed{\sigma(x) = \rho g x - \rho g l + t_L}$$

Alternatively:

$$\sigma(l) - \sigma(x) = \int_x^l \rho g \, dx = \rho g (l - x) \quad \left(\text{Integrating } \int_x^l \bullet\right)$$

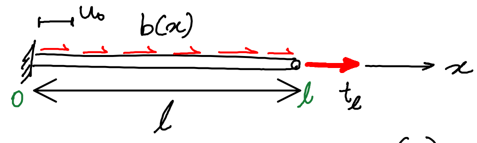
$$\Rightarrow \boxed{\sigma(x) = \rho g x - \rho g l + t_L} \quad \left(t_L = \sigma(l)(+1)\right)$$

For displacement:

$$C u' = \sigma(x) = \rho g x - \rho g l + t_L$$

$$\Rightarrow \boxed{u(x) = \frac{1}{C}\left[\rho g \frac{x^2}{2} - \rho g l x + t_L x\right]} + \cancel{c_2}^{\,u_0 = 0} \quad \overset{BC}{\left(u(0) = u_0 = 0\right)}$$

Strong Forms and Weak forms



$\sigma = \sigma(\epsilon)$

Consider the 1-D problem:

Find $\sigma(x)$, $u(x)$:

$$\text{(S)}\left[\begin{array}{lll} \text{GDE}\left[\ \sigma' + b = 0\right. & & \forall x \in (0,\ell) \\ \text{BC}\left[\begin{array}{l} u(0) = u_0 \\ \sigma(\ell)(+1) = t_\ell \end{array}\right. & & \begin{array}{l} \text{on } \Gamma_D \\ \text{on } \Gamma_N \end{array} \end{array}\right.$$

{ Residual: $g(x) = \sigma' + b$

This is called the Strong Form (S) of the governing differential equation (GDE).

Weak forms

Define $\quad G(\sigma,\bar u) \equiv \ominus\left[\int_0^\ell \bar u(x)\, g(x)\, dx + \bar u(0)\big(t_0 + \sigma(0)\big) + \bar u(\ell)\big(t_\ell - \sigma(\ell)\big)\right]$

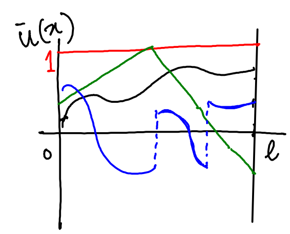
$G(\sigma,\bar u)$: A scalar functional (function of functions)

Example: σ $\bar u$

$$G\left(\tfrac{1}{2}x^2, x\right) = -\int_0^\ell x \cdot \overset{(x+b)}{\cancel{x}}\, dx - 0\,(t_0 + 0) - \ell\,(t_\ell - \ell) = \underbrace{-\frac{\ell^3}{3} - \ell\,(t_\ell - \ell) + \frac{b\ell^2}{2}}_{\text{scalar}}$$

Input functions $\qquad g = \sigma' + b = (x + b)$

Weak form of the Problem Statement.



If for some $\sigma(x)$, t_0

$$G(\sigma,\bar u) = 0 \quad \forall\ \bar u \in V(0,\ell)$$

for all $\qquad$ Function space

Then $\Rightarrow$

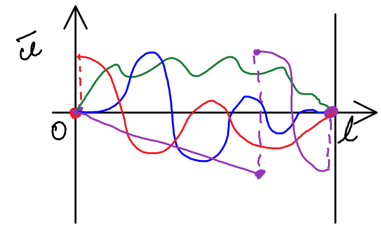
$$\begin{array}{lll} g = \sigma' + b = 0 & & \forall x \text{ in } (0,\ell) \\ \sigma(\ell)(+1) = t_\ell & & \text{at } x = \ell \\ \sigma(0)(-1) = t_0 & & \text{at } x = 0 \end{array}$$

This relies on the fundamental theorem of Calculus of Variations:

$$\text{If } G(\sigma,\bar u) = \ominus\left[\int_0^\ell \bar u(x)\, \overbrace{g(x)}^{\sigma' + b}\, dx - \bar u(0)\,\overbrace{a_0}^{t_0 + \sigma(0)} - \bar u(\ell)\,\overbrace{a_\ell}^{t_\ell - \sigma(\ell)}\right] = 0 \quad \forall\ \bar u(x) \in V(0,\ell)$$

Then $\Rightarrow \quad g(x) = 0 \quad$ and $\quad a_0 = 0 \quad$ and $\quad a_\ell = 0$

Proof of Fundamental theorem of calculus of variations

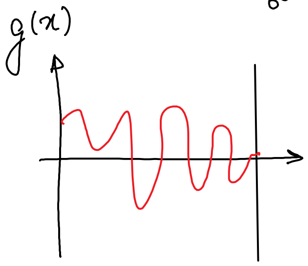


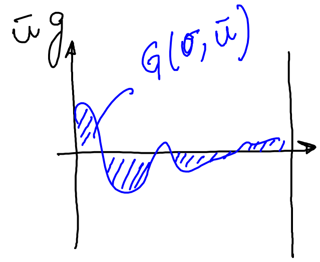
First, let $\bar{u}(0) = \bar{u}(\ell) = 0$

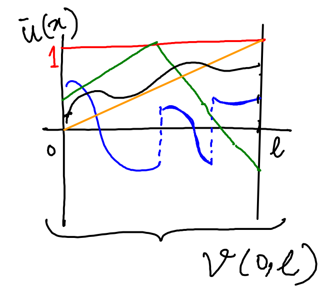
There exists $\bar{u}(x) = g(x) \in \mathcal{V}(0,\ell)$

So if $G(\sigma,\bar{u}) = -\int_0^\ell \bar{u}\, g\, dx$

$$= -\int_0^\ell g^2\, dx = 0 \Rightarrow g = 0 \Rightarrow \sigma' + b = 0 \quad \forall x \in (0,\ell)$$







Now in addition if $\bar{u}(0), \bar{u}(\ell) \neq 0 \Rightarrow$

If $G(\sigma,\bar{u}) = -\int_0^\ell \bar{u}\, g\, dx - \bar{u}(0)\left(t_0 + \sigma(0)\right) - \bar{u}(\ell)\left(t_\ell - \sigma(\ell)\right) = 0 \quad \forall \bar{u} \in \mathcal{V}(0,\ell)$

$$\Rightarrow \quad t_0 + \sigma(0) = 0 \quad ; \quad t_\ell - \sigma(\ell) = 0$$

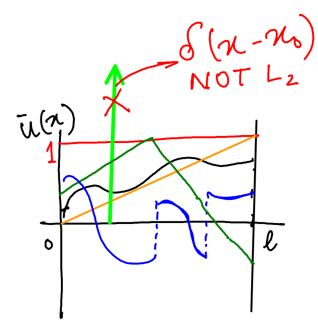
Restriction on the choice of Function spaces $\mathcal{V}(0,\ell)$

Fundamental theorem of Calculus of variations restricts $\bar{u} \in \mathcal{V}(0,\ell)$:

$\bar{u}$ must be square integrable

ie $\underbrace{\int_0^\ell (\bar{u})^2\, dx}_{L_2\text{-norm of } \bar{u}(x)}$ must exist (be finite)

(L: Lebesgue)



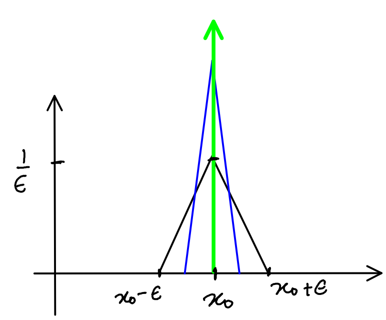
Note: Dirac-delta $\delta(x-x_0)$ is not L_2.

$$\int_{-\infty}^{\infty} \delta(x - x_0)\, dx = 1$$

$$\int_{-\infty}^{\infty} \left[\delta(x-x_0)\right]^2 dx = \frac{2}{3\epsilon} \longrightarrow \infty$$

(as $\epsilon \to 0$)

$\lim_{\epsilon \to 0}$



Possible choices for function spaces:

- Square integrable : L_2 (or H^0)
- Sq. Int. upto 1^{st} derivative : H^1
- Sq. Int upto m^{th} derivative : H^m

- Continuous functions : C^0
- Continuous upto 1^{st} der. : C^1
- Continuous upto m^{th} der : C^m

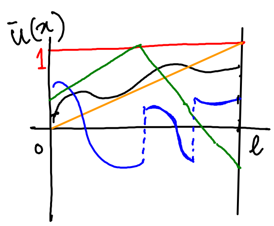
C^∞ smooth (polynomial, Trigs, Exp)

$$C^\infty \subset C^m \subset H^m \subset \dots C^1 \subset H^1 \subset C^0 \subset (H^0 = L_2)$$

subset

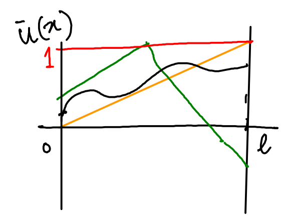
Examples of function spaces:

L_2 / H^0:



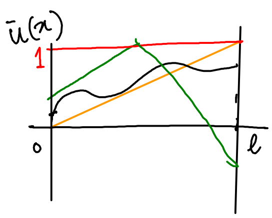
(discontinuous functions are possible)

C^0



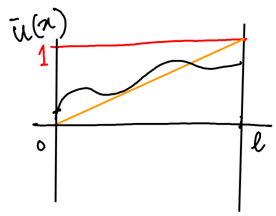
continuous functions only

H^1:



(first derivatives square integrable)

C^1

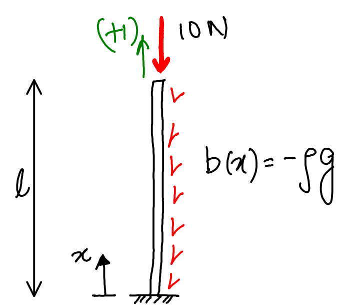


(first derivative continuous)

Example of Approximate solution to Weak form:

Recall exact solution:

$$\sigma(x) = \rho g x - \rho g \ell + t_L$$



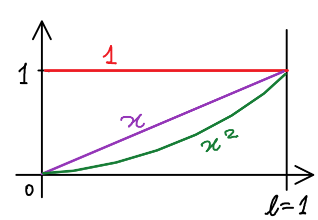
(W) [Find σ, t_0 such that

$$G(\sigma, \bar{u}) = 0 \qquad \forall\ \bar{u} \in V(0,1)$$

Let $\sigma = a_0 + a_1 x + a_2 x^2$

and $\bar{u} = \bar{a}_0 1 + \bar{a}_1 x + \bar{a}_2 x^2$

a_i : Unknown
$\bar{a}_i$: Arbitrary



For $G(\sigma, u) = 0$

$$\Rightarrow\ -\int_0^1 (\bar{a}_i \bar{h}_i(x)) \cdot (a_j h_j(x))\, dx - \bar{u}(0)(t_0 + \sigma(0)) - \bar{u}(\ell)\,(t_\ell - \sigma(\ell)) = 0$$

$$\Rightarrow\ \underline{\bar{a}}\,[\underset{\sim}{K}\,\underline{a} - \underline{f}] = 0 \qquad \forall\ \bar{a}_i \in \mathbb{R}$$

Weak Form: Method of weighted Residuals (MWR):

MWR (W)

$$\boxed{G(\sigma,\bar{u}) \equiv -\int_0^\ell \bar{u}(x) g(x)\,dx - \bar{u}(0)(t_0 + \sigma(0)) + \bar{u}(\ell)(\sigma(\ell) - t_\ell)}$$

$$G(\sigma,\bar{u}) = -\underbrace{\int_0^\ell \bar{u}(\sigma' + b)\,dx}_{(Cu')'} - \bar{u}(0)(\sigma(0)+t_0) + \bar{u}(\ell)(\sigma(\ell) - t_\ell)$$

$$= -\int_0^\ell d(\bar{u}\sigma) + \int_0^\ell \bar{u}'\sigma\,dx - \int_0^\ell \bar{u} b\,dx$$

$(\bar{u}\sigma)' = \bar{u}'\sigma + \bar{u}\sigma'$

$\Rightarrow -\bar{u}\sigma' = -\dfrac{d(\bar{u}\sigma)}{dx} + \bar{u}'\sigma$

(Integration by parts)

$$= -[\bar{u}\sigma]_0^\ell + \int_0^\ell \bar{u}'\sigma\,dx - \int_0^\ell \bar{u} b\,dx$$

$$= \cancel{\bar{u}(0)\,\sigma(0)} - \cancel{\bar{u}(\ell)\,\sigma(\ell)} + \int_0^\ell \bar{u}'\sigma\,dx - \int_0^\ell \bar{u} b\,dx$$

$$- \bar{u}(0)(t_0 + \cancel{\sigma(0)}) + u(\ell)(\cancel{\sigma(\ell)} - t_\ell)$$

Unknown reactions

Applied traction ✓

Weak Form: Principle of Virtual Work (PVW):

PVW (W) $\Rightarrow$

$$\boxed{G(\sigma,\bar{u}) = \underbrace{\int_0^\ell (\bar{u}'\sigma)\,dx}_{W_I} \underbrace{- \int_0^\ell \bar{u} b\,dx - t_\ell\,\bar{u}(\ell) - t_0\,\bar{u}(0)}_{-W_E}}$$

Virtual Strain $\bar{u}' = \bar{\epsilon}$

Internal Virtual Work

External Virtual Work.

Substituting the material model: $\sigma = Cu'$

Weak form problem statements:

MWR (W)

> Find $u(x) \in H^2(0,\ell)$ and t_0 such that $u(0) = u_0$
>
> $G(u,\bar{u}) = 0 \quad \forall \quad \bar{u}(x) \in L_2(0,\ell) = H^0(0,\ell)$
>
> where $G(u,\bar{u}) \equiv -\int_0^\ell \bar{u}((Cu')' + b)\,dx - \bar{u}(0)(t_0 + \sigma(0)) - \bar{u}(\ell)(t_\ell - \sigma(\ell))$

OR

PVW (W)

> Find $u(x) \in H^1(0,\ell)$ and t_0 such that $u(0) = u_0$
>
> $G(u,\bar{u}) = 0 \quad \forall \quad \bar{u}(x) \in \cancel{L_2(0,\ell)}\ H^1(0,\ell)$
>
> where $G(u,\bar{u}) \equiv \int_0^\ell \bar{u}' C u'\,dx - \int_0^\ell \bar{u} b\,dx - \bar{u}(0)\,t_0 - \bar{u}(\ell)\,t_\ell$

Boundary Conditions

In the weak form of the problem statement:

MWR (W):

$$\begin{cases} \text{Find } t_0 \text{ and } u(x) \in H^2(0,\ell) \text{ such that } \boxed{u(0) = u_0} \\ \text{and } G(\sigma, \bar{u}) = 0 \quad \forall \quad \bar{u}(x) \in H^0(0,\ell) \\ \text{where } G(\sigma,\bar{u}) \equiv -\int_0^\ell \bar{u}\,(\sigma' + b)\,dx - \bar{u}(0)\,(t_0 + \sigma(0)) - \bar{u}(\ell)\boxed{(t_\ell - \sigma(\ell))} \end{cases}$$

ESSENTIAL BC ($\Gamma_D : x = 0$)

NATURAL BC ($\Gamma_N : x = \ell$)

OR

PVW (W):

$$\begin{cases} \text{Find } t_0 \text{ and } u(x) \in H^1(0,\ell) \text{ such that } \boxed{u(0) = u_0} \\ \text{and } G(\sigma, \bar{u}) = 0 \quad \forall \quad \bar{u}(x) \in H^1(0,\ell) \\ \text{where } G(\sigma,\bar{u}) \equiv \underbrace{\int_0^\ell \bar{u}'\,\sigma\,dx}_{W_I} - \underbrace{\left[\int_0^\ell \bar{u}\,b\,dx + \bar{u}(0)\,t_0 + \bar{u}(\ell)\,t_\ell\right]}_{W_E} \end{cases}$$

ESSENTIAL BC

Unknowns to solve for: $\sigma(x)$ & t_0 (or $u(x)$ & t_0 where $\sigma = \sigma(u')$)

But $t_0 = \sigma(0)\,(-1)$, so we can eliminate t_0.

By further restricting $\boxed{\bar{u} = 0 \text{ on } \Gamma_D}$ (even if actual $u_0 \neq 0$)

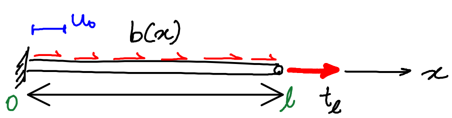
HOMOGENEOUS ESSENTIAL BC.

Define the spaces:
(for virtual displacement)

$H_E^0(0,\ell) \equiv H^0(0,\ell)$ and $\bar{u} = 0$ on Γ_D

$H_E^1(0,\ell) \equiv H^1(0,\ell)$ and $\bar{u} = 0$ on Γ_D

Thus the final weak forms are:



space for real displacement

MWR (W):

$$\begin{cases} \text{Find } \cancel{t_0 \text{ and }} u(x) \in H^2(0,\ell) \text{ such that } \boxed{u(0) = u_0} \\ \text{and } G(\sigma, \bar{u}) = 0 \quad \forall \quad \bar{u}(x) \in H_E^0(0,\ell) \\ \text{where } G(\sigma,\bar{u}) \equiv -\int_0^\ell \bar{u}\,(\sigma' + b)\,dx - \bar{u}(\ell)\boxed{(t_\ell - \sigma(\ell))} \end{cases}$$

EBC ($\Gamma_D : x = 0$)

space for virtual disp. (HEBC)

NBC ($\Gamma_N : x = \ell$)

OR

space for real displacement

PVW (W):

$$\begin{cases} \text{Find } \cancel{t_0 \text{ and }} u(x) \in H^1(0,\ell) \text{ such that } \boxed{u(0) = u_0} \\ \text{and } G(\sigma, \bar{u}) = 0 \quad \forall \quad \bar{u}(x) \in H_E^1(0,\ell) \\ \text{where } G(\sigma,\bar{u}) \equiv \underbrace{\int_0^\ell \bar{u}'\,\sigma\,dx}_{W_I} - \underbrace{\left[\int_0^\ell \bar{u}\,b\,dx + \bar{u}(\ell)\,t_\ell\right]}_{W_E} \end{cases}$$

EBC

space for virtual disp (HEBC)

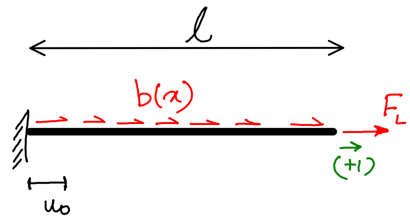
Note: (S) $\Leftrightarrow$ (W)

(S) Strong form $\quad (cu')' + b = 0 \quad \forall x \in (0, \ell)$ $\quad$ (assume $A = 1$)

(+ some BCs)

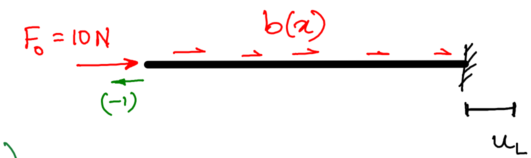
(1)
$u(x=0) = u_0$ $\quad$ EBC $\Rightarrow \bar{u}(0) = 0$ (HEBC)

$\sigma(\ell)(+1) = t_L = \dfrac{F_L}{A}$ $\quad$ NBC



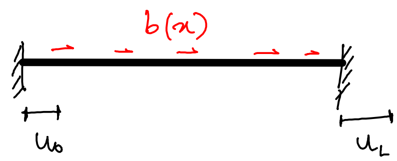
(2)
$\sigma(0)(-1) = t_0 = \dfrac{F_0}{A}$ $\quad$ NBC

$u(x=\ell) = u_L$ $\quad$ EBC $\Rightarrow \bar{u}(\ell) = 0$ (HEBC)



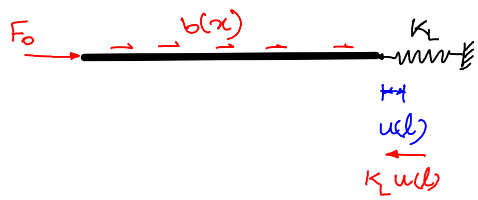
(3)
$u(x=0) = u_0$ $\quad$ EBC $\Rightarrow \bar{u}(0) = 0$ (HEBC)

$u(x=\ell) = u_L$ $\quad$ EBC $\Rightarrow \bar{u}(\ell) = 0$ (HEBC)



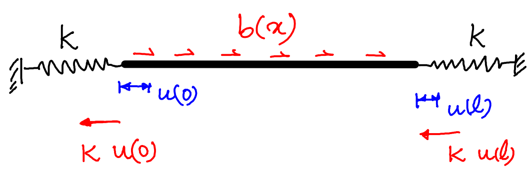
(4)
$\sigma(0)(-1) = t_0 = \dfrac{F_0}{A}$ $\quad$ NBC

$\sigma(\ell)(+1) = t_L = \dfrac{F_L}{A} = \dfrac{-K\,u(\ell)}{A}$ $\quad$ Mixed BC

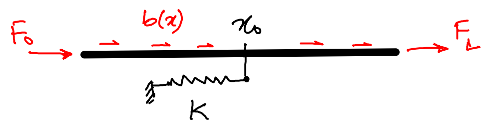


(5)
$\sigma(0)(-1) = t_0 = \dfrac{-K\,u(0)}{A}$ (Mixed)

$\sigma(\ell)(+1) = t_\ell = \dfrac{-K\,u(\ell)}{A}$ (Mixed)



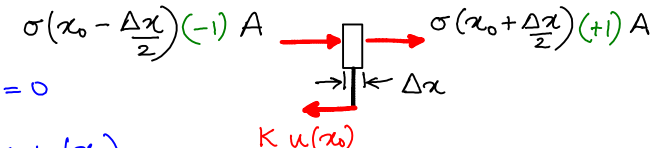
(6)
$\left.\begin{array}{l} \sigma(0)(-1) = t_0 = \dfrac{F_0}{A} \\ \sigma(\ell)(+1) = t_L = \dfrac{F_L}{A} \end{array}\right\}$ NBCs



Jump condition

$$\sigma(x_0)\big|_+ - \sigma(x_0)_- - K\,u(x_0) = 0$$

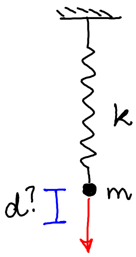
$$\Rightarrow [\![\sigma(x_0)]\!] = K\,u(x_0)$$



Variational (Energy-based) methods

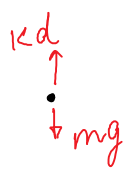
In contrast with Weighted residuals, sometimes it is possible to derive the weak form from Variational (Energy) Principles. {Ref. Hjelmstad ch 9

Example



Equilibrium (MWR/PVW)

(FBD)



$\Rightarrow kd = mg$

$\Rightarrow \boxed{d = \dfrac{mg}{k}}$

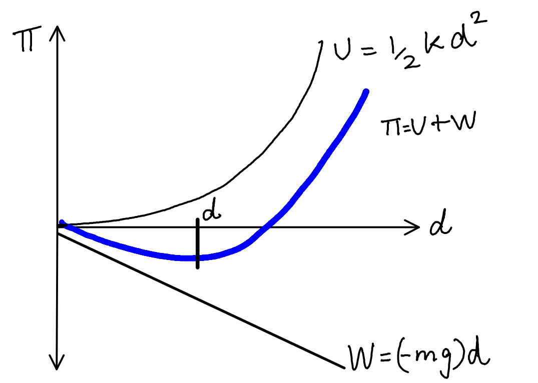
Minimization of Potential Energy

$$\Pi = U + W$$

$$\Pi = \tfrac{1}{2}kd^2 + (-mgd)$$

Minimum $\Rightarrow \dfrac{\partial \Pi}{\partial d} = 0 \Rightarrow kd - mg = 0$

$\Rightarrow kd = mg$ (Equilibrium)



Now lets consider:

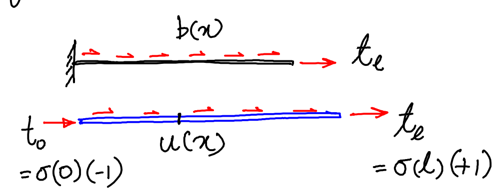
$$Cu'' + b = 0 \qquad \text{for } x \in (0, \ell)$$

$$u(0) = u_0 \quad ; \quad Cu'(\ell) = t_\ell$$

Total Potential Energy

$$\Pi = U + W$$

U: Strain energy; W: Potential due to conservative body forces & external tractions



Recall

$$U(u) = \int_0^\ell \underbrace{\tfrac{1}{2}\sigma\epsilon}_{\psi(\epsilon)} \, dx = \int_0^\ell \tfrac{1}{2} C(u')^2 \, dx$$

$$W(u) = \int_0^\ell -ub \, dx \; - u(\ell)\underbrace{t_\ell}_{\sigma(\ell)(+1)} - u(0)\underbrace{t_0}_{\sigma(0)(-1)}$$

Thus
$$\boxed{\Pi(u) = \int_0^\ell \left[\tfrac{1}{2}C(u')^2 - ub\right] dx - u(\ell)\,t_\ell - u(0)\,t_0}$$

Note: Real displacement only

To minimize Potential Energy $\Pi(u)$ wrt $u(x)$ we need: Directional Derivative (Gateaux Derivative)

$$\boxed{D\,\Pi(u)\cdot\bar{u} = \left[\frac{d}{d\epsilon}\left(\Pi(u + \epsilon\bar{u})\right)\right]\Bigg|_{\epsilon=0} = 0}$$

Energy weak form:

$$D\,\Pi(u)\cdot\bar u = \left[\frac{d}{d\epsilon}\left[\Pi(u+\epsilon\bar u)\right]\right]\Bigg|_{\epsilon=0} = \frac{d}{d\epsilon}\Bigg[\int_0^{\ell} \tfrac{1}{2}\, c\,(u'+\epsilon\bar u')^2 dx - \int_0^{\ell}(u+\epsilon\bar u)\, b\, dx$$

$$- (u(0)+\epsilon\bar u(0))\,t_0 - (u(\ell)+\epsilon\bar u(\ell))\,t_\ell\Bigg]$$

E

$$= \frac{d}{d\epsilon}\Bigg[\int_0^{\ell} \tfrac{1}{2} C\big(\cancel{u'^2}_{0} + 2\cancel{\epsilon}_{1}\, u'\bar u' + \cancel{\epsilon^2}_{2\epsilon\to 0}\,\bar u'^2\big)\, dx - \int_0^{\ell}(\cancel{u}_{0}+\cancel{\epsilon}_{1}\bar u)\, b\, dx$$

$$-\left[\cancel{u(0)}_{0} + \cancel{\epsilon}_{1}\,\bar u(0)\right] t_0 - \left(\cancel{u(\ell)}_{0} + \cancel{\epsilon}_{1}\,\bar u(\ell)\right) t_\ell\Bigg]$$

i.e.

$$= \underbrace{\int_0^{\ell} C u'\bar u'\, dx}_{W_I} - \underbrace{\int_0^{\ell} \bar u\, b\, dx - \bar u(\ell)\, t_\ell - \bar u(0)\, t_0}_{W_E} = G(u,\bar u)\ !$$

$$\boxed{D\,\Pi(u)\cdot\bar u = G(u,\bar u)}$$

E $\Rightarrow$ W $\qquad$ W $\boxed{\text{PVW}}$

Integrate by parts (in reverse – to unbalance the derivatives):

$$= -\int_0^{\ell} \bar u\, C u''\, dx + \left[\bar u\, C u'\right]_0^{\ell} - \int_0^{\ell} \bar u\, b\, dx - \bar u(\ell)\, t_\ell - u(0)\, t_0$$

HEBC

$$= -\int_0^{\ell} \bar u\, \underbrace{(Cu''+b)}_{S}\, dx + \bar u(\ell)\underbrace{\left[C u'(\ell) - t_\ell\right]}_{\text{Natural BC @ } \ell} - \cancel{\bar u(0)}^{\,0}\underbrace{\left[C u'(0) + t_0\right]}_{\text{Nat BC @ } 0}$$

Now if

$$D\Pi(u)\cdot\bar u = G(u,\bar u) = 0 \qquad \text{for } \underline{\text{all}}:\ \forall\ \bar u(x) \in H_0^1(0,\ell)$$

then

S $\left[\begin{array}{ll} Cu''+b = 0 & \text{at all points } x \text{ in } (0,\ell) \\ \text{and} & \\ Cu'(\ell) = \sigma(\ell) = t_\ell & \text{at } x=\ell \end{array}\right.$

i.e. we get the governing differential equation (GDE) back from the variational principle.

In general, if you have "some" $\boxed{\Pi(u)}$
i.e. some "energy" functional

then, the governing differential equation corresponding to $\Pi(u)$ is called its <u>Euler equation</u> (or <u>Euler-Lagrange</u> equation) S

Vainberg's Theorem: Existence of an Energy form (Variational form)

$$\Pi(u) \xrightarrow{\text{Min wrt } \bar{u}} D\Pi(u)\cdot\bar{u} = G(u,\bar{u})=0 \quad \forall \bar{u} \xrightarrow[\text{FTCV}]{\text{RIBP}} \begin{array}{l}(Cu')'+b=0\\ \sigma(l) = t_e\end{array}$$

E $\qquad\qquad\qquad$ W $\qquad$ PVW $\qquad\qquad$ S

$$\boxed{\Pi(u) \xleftarrow{\boxed{?}} D\Pi(u)\cdot\bar{u}} \;\boxed{?}\; G(u,\bar{u})=0 \quad \forall\bar{u} \xleftarrow[\text{IBP}]{\text{MWR}} \begin{array}{l}A(u) = f\\ \text{(some GDE)}\end{array}$$

Existence of a Variational Principle is decided with the help of the Vainberg's Theorem:

$\tilde{G}^h(\underline{d},\bar{\underline{d}})$ Approximate solutions

Vainberg's Theorem

Given a functional $G(u,\bar{u})$ {i.e PVW W or GDE strong form S}

If
- $G(u,\bar{u})$ is linear in the second argument:

  ie $$G(u,(\alpha\bar{u}_1+\beta\bar{u}_2)) = \alpha\, G(u,\bar{u}_1)+\beta\, G(u,\bar{u}_2)$$

- Directional derivative is symmetric in the second argument:

  ie $$DG(u,\bar{u}_1)\cdot\bar{u}_2 = DG(u,\bar{u}_2)\cdot\bar{u}_1$$

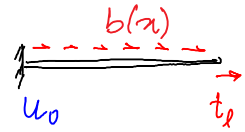
$$\left\{\text{where } DG(u,\bar{u}_1)\cdot\bar{u}_2 \equiv \left[\frac{d}{d\epsilon}G\big((u+\epsilon\bar{u}_2),\bar{u}_1\big)\right]\Bigg|_{\epsilon=0}\right\}$$

Then

$$\boxed{\Pi(u) = \int_0^1 G(tu,u)\,dt + \underset{0}{\cancel{c}}}$$ such that $D\Pi(u)\cdot\bar{u} = G(u,\bar{u})$

Example:

Consider the Weak form for the 1-D problem:-



(W) $$G(u,\bar{u}) = \int_0^\ell \bar{u}' C u' \, dx - \int_0^\ell \bar{u} b \, dx - \bar{u}(\ell)\, t_\ell$$

Vainberg's Th:

- Linearity of $\bar{u}$ ✓
- Symmetric in $\bar{u}$ ✓ $\quad DG(u,\bar{u}_1)\cdot\bar{u}_2 \stackrel{?}{=} DG(u,\bar{u}_2)\cdot\bar{u}_1$

LHS

$$DG(u,\bar{u}_1)\cdot\bar{u}_2 = \frac{d}{d\epsilon}\left[\int_0^\ell \bar{u}_1' C(\underbrace{\cancel{u}}_{0}+\underbrace{\cancel{\epsilon}}_{1}\bar{u}_2)' \, dx - \int_0^\ell \cancel{\bar{u}_1 b} \, dx - \cancel{\bar{u}_1(\ell)\, t_\ell}\right]_{\epsilon=0}$$

$$= \int_0^\ell \bar{u}_1' C \bar{u}_2' \, dx \quad (*)$$

RHS

$$DG(u,\bar{u}_2)\cdot\bar{u}_1 = \frac{d}{d\epsilon}\left[\int_0^\ell \bar{u}_2' C(\cancel{u}+\underbrace{\cancel{\epsilon}}_{1}\bar{u}_1)' \, dx - \int_0^\ell \cancel{\bar{u}_2 b} \, dx - \cancel{\bar{u}_2(\ell)\, t_\ell}\right]_{\epsilon=0}$$

$$= \int_0^\ell \bar{u}_2' C \bar{u}_1' \, dx \quad (*)$$

$\Rightarrow \Pi(u)$ exists!

$$\int_0^1 t\, dt = \left[\frac{t^2}{2}\right]_0^1 = 1/2$$

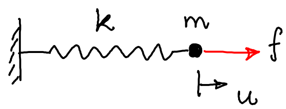
$$\Pi(u) = \int_0^1 G(tu,u)\, dt + \overset{0}{\cancel{c}}$$

$$= \int_0^1 \left[\int_0^\ell u' C (tu)' \, dx - \int_0^\ell u b \, dx - u(\ell)\, t_\ell\right] dt$$

$$\Rightarrow \Pi(u) = \int_0^\ell \tfrac{1}{2} C(u')^2 \, dx - \int_0^\ell u b \, dx - u(\ell)\, t_\ell \quad (E)$$

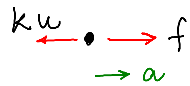
Hamilton's Principle (for Dynamics)

Consider



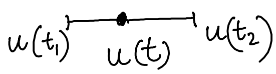
Equation of motion: (Dynamic Equilibrium $F = ma$)

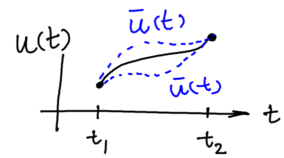
FBD:



$$m\ddot{u} + ku - f = 0 \qquad \forall t$$

Using Energy principles (Variational methods):





$$K(\dot{u}) = \tfrac{1}{2} m \dot{u}^2 \quad ; \quad \Pi(u) = \tfrac{1}{2} k u^2 - f.u$$

Define: Lagrangian $\boxed{\mathcal{L}(u, \dot{u}) = K(\dot{u}) - \Pi(u)}$

Hamiltonian

$$H(u, p) = \underbrace{K(p)}_{\frac{1}{2}\frac{p^2}{m}} + \Pi(u), \qquad p \to m\dot{u}$$

Hamilton's Principle:

$$D\Big[\overbrace{\int_{t_1}^{t_2} \mathcal{L}(u, \dot{u})\, dt}^{\text{Action}}\Big] \cdot \bar{u} = 0 \qquad \forall\, \bar{u}(t)$$

$$\Rightarrow \frac{d}{d\epsilon}\left[\int_{t_1}^{t_2} \left[\tfrac{1}{2} m (\dot{u} + \epsilon \dot{\bar{u}})^2 - \tfrac{1}{2} k (u + \epsilon \bar{u})^2 + f.(u + \epsilon \bar{u})\right] dt\right]\Bigg|_{\epsilon = 0} = 0$$

$$\Rightarrow \int_{t_1}^{t_2} \left(m \dot{u} \dot{\bar{u}} - k u \bar{u} + f \bar{u}\right) dt = 0 \qquad \forall\, \bar{u}(t)$$

(Integrate by parts in t)

$$\Rightarrow \int_{t_1}^{t_2} \left(-\bar{u}\, m\ddot{u} - k u \bar{u} + f \bar{u}\right) dt + \Big[\cancel{m \dot{u} \bar{u}}\Big]_{t_1}^{t_2} = 0 \qquad \forall\, \bar{u}(t)$$

$$\left(\bar{u}(t_1) = \bar{u}(t_2) = 0\right)$$

$$\Rightarrow \boxed{m\ddot{u} + ku - f = 0}$$

This is the Euler-Lagrange equation corresponding to the Lagrangian above.

Using the same Lagrangian $\mathcal{L}(u, \dot{u}) = K(\dot{u}) - \Pi(u)$

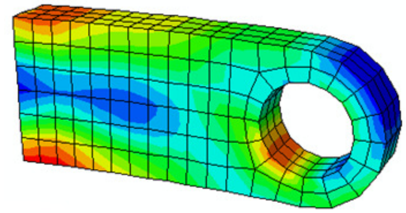
For 1-D problem:

$$\sigma' + b = \rho \ddot{u}$$

For 2-D & 3D problems:

$$\operatorname{div} \underset{\sim}{\sigma} + \underline{b} = \rho \underline{\ddot{u}}$$

Weak form in 3D



MWR (W)

$$G(\underline{u},\bar{\underline{u}}) \equiv -\int_\Omega \bar{\underline{u}}\cdot\big(\mathrm{div}\,\overset{\mathbb{C}\boldsymbol{\epsilon}}{\boldsymbol{S}} + \underline{b}\big)\,d\Omega - \int_{\Gamma_N} \bar{u}_i\big(t_{(n)i} - S_{ij}n_j\big)\,d\Gamma$$

$$= -\int_\Omega \bar{u}_i\big[(S_{ij,j}) + b_i\big]\,d\Omega - \int_{\Gamma_N} \bar{u}_i\big(t_{(n)i} - S_{ij}n_j\big)\,d\Gamma$$

(Product Rule)

Note: $(\bar{u}_i S_{ij})_{,j} = \bar{u}_{i,j} S_{ij} + \bar{u}_i S_{ij,j}$

$$-\bar{u}_i S_{ij,j} = -(\bar{u}_i S_{ij})_{,j} + \bar{u}_{i,j} S_{ij}$$

$$\Rightarrow G(\underline{u},\bar{\underline{u}}) = \int_\Omega \overset{\nabla\bar{\underline{u}}}{\bar{u}_{i,j}}\, S_{ij}\,d\Omega - \int_\Omega \bar{u}_i b_i\,d\Omega - \int_\Omega \underbrace{(\bar{u}_i S_{ij})_{,j}}\,d\Omega - \int_{\Gamma_N} \bar{u}_i\big(t_{(n)i} - S_{ij}n_j\big)\,d\Gamma$$

Divergence Theorem: $-\int_\Omega \mathrm{div}(\boldsymbol{S}^T\bar{\underline{u}})\,d\Omega = -\int_\Gamma \underbrace{(\boldsymbol{S}^T\bar{\underline{u}})\cdot\underline{n}}\,d\Gamma$

$$G(\underline{u},\bar{\underline{u}}) = \int_\Omega \underbrace{\bar{u}_{i,j} S_{ij}}_{\bar{\boldsymbol{\epsilon}}:\boldsymbol{S}}\,d\Omega - \int_\Omega \underbrace{\bar{u}_i b_i}_{\bar{\underline{u}}\cdot\underline{b}}\,d\Omega - \int_\Gamma (\bar{u}_i S_{ij})n_j\,d\Gamma - \int_{\Gamma_N} \bar{u}_i\big(\underbrace{t_{(n)i} - \cancel{S_{ij}n_j}}\big)\,d\Gamma$$

$$\underline{t}_{(n)} = \boldsymbol{S}\underline{n}$$

Note:

$$\bar{u}_{i,j} = \underbrace{\tfrac{1}{2}(\bar{u}_{i,j} + \bar{u}_{j,i})}_{\text{sym}\,\nabla\bar{\underline{u}}\ (=\bar{\boldsymbol{\epsilon}})} + \underbrace{\tfrac{1}{2}(\bar{u}_{i,j} - \bar{u}_{j,i})}_{\text{skew}\,\nabla\bar{\underline{u}}}$$

$$-\int_{\Gamma_D} \underset{0}{\cancel{\bar{u}_i}}\, t_{(n)i}\,d\Gamma - \cancel{\int_{\Gamma_N} \bar{u}_i S_{ij} n_j\,d\Gamma}$$

(HEBC)

$$\bar{u}_{i,j}\,S_{ij} = \bar{\epsilon}_{ij} S_{ij} + \cancel{\tfrac{1}{2}(\bar{u}_{i,j} - \bar{u}_{j,i})S_{ij}} = \bar{\boldsymbol{\epsilon}}:\boldsymbol{S} + \cancel{\text{skew}(\nabla\bar{\underline{u}}):\boldsymbol{S}}$$

$$\Rightarrow \boxed{G(\underline{u},\underline{u}) = \underbrace{\int_\Omega \bar{\boldsymbol{\epsilon}}:\boldsymbol{S}\,d\Omega}_{W_I}\ \underbrace{-\int_\Omega \bar{\underline{u}}\cdot\underline{b}\,d\Omega - \int_{\Gamma_N} \bar{\underline{u}}\cdot\underline{t}_{(n)}\,d\Gamma}_{-W_E}}$$

(PVW) (W)

Energy form in 3D

One can show that the Vainberg's Theorem is satisfied for the above weak form.

$$\Rightarrow \pi(\underline{u}) = \int_0^1 G(t\underline{u},\underline{u})\,dt$$

$$= \int_0^1 \left[\int_\Omega \boldsymbol{\epsilon}:\mathbb{C}(t\boldsymbol{\epsilon})\,d\Omega - \int_\Omega \underline{u}\cdot\underline{b}\,d\Omega - \int_{\Gamma_N} \underline{u}\cdot\underline{t}_{(n)}\,d\Gamma\right] dt$$

$$\Rightarrow \boxed{\pi(\underline{u}) = \int_\Omega \tfrac{1}{2}\boldsymbol{\epsilon}:\mathbb{C}\,\boldsymbol{\epsilon}\,d\Omega - \int_\Omega \underline{u}\cdot\underline{b}\,d\Omega - \int_{\Gamma_N} \underline{u}\cdot\underline{t}_{(n)}\,d\Gamma}$$