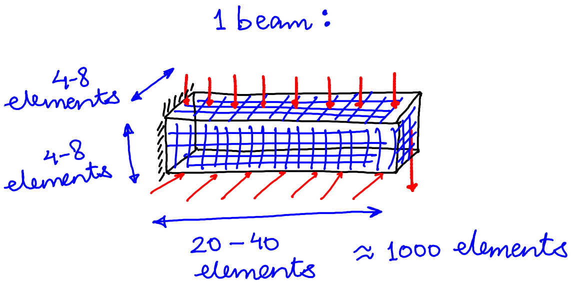
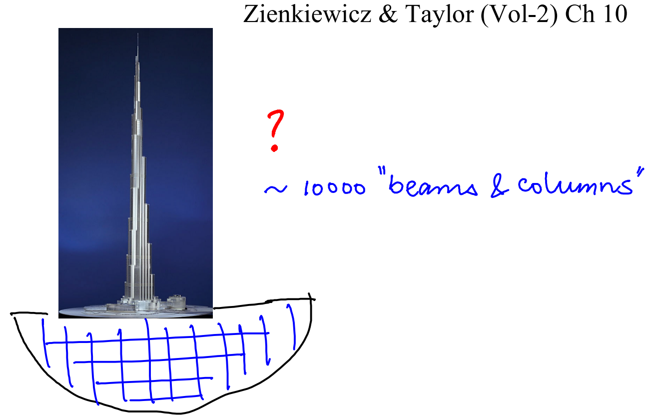
Chapter 4: Beams and Frames

Ref: Hjelmstad Ch 7
Reddy Ch 5
Hughes Section 5.4
Zienkiewicz & Taylor (Vol-2) Ch 10

- Need for "Structural" Theories?





- Governing Equations for Solid / Continuum mechanics:

Find $\underline{\phi}(\underline{X}) \quad (= \underline{X} + \underline{u}(\underline{X}))$ such that

$$(\textstyle\sum \underline{F} = \underline{0}) \Rightarrow \operatorname{div} \underset{\sim}{\sigma} + \underline{b} = \underline{0} \qquad \text{everywhere in } \text{"}\Omega\text{"}$$

$$(\textstyle\sum \underline{M} = \underline{0}) \Rightarrow \underset{\sim}{\sigma} = \underset{\sim}{\sigma}^T \qquad \text{"}$$

BCs:

$$\underline{u} = \underline{u}_0 \qquad \text{on } \Gamma_D$$

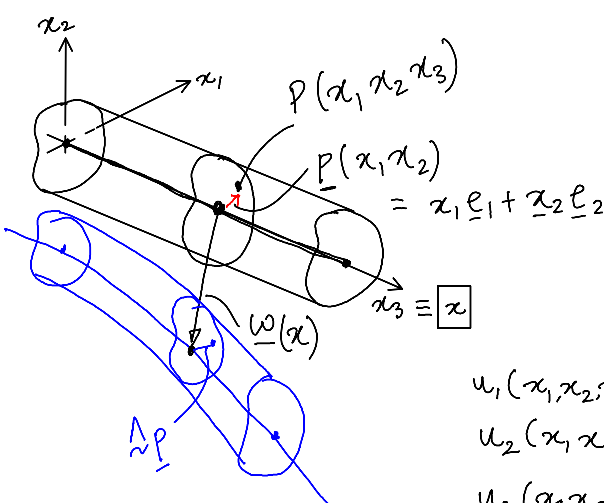
$$\underset{\sim}{\sigma}\,\underline{n} = \underline{t}_n \qquad \text{on } \Gamma_N$$

strain-displacement: $\underset{\sim}{\varepsilon} = \frac{1}{2}\left(\nabla \underline{u} + \nabla \underline{u}^T\right)$ everywhere in Ω

Constitutive: $\underset{\sim}{\sigma} = \underset{\sim}{\sigma}(\underset{\sim}{\varepsilon})$ "

$$\left(\underset{\sim}{\sigma} = \lambda \, tr(\underset{\sim}{\varepsilon})\, \underset{\sim}{I} + 2\mu\, \underset{\sim}{\varepsilon}\right)$$

- Fundamental Kinematic Assumption of Beam Theory:



$$\underset{(x_1 x_2 x_3)}{\underline{\phi}(\underline{X})} = x_3 \underline{e}_3 + \underline{\omega}(x_3) + \underbrace{\underset{\sim}{\Lambda}\,\underline{p}}_{\underline{p} + \underline{\theta} \times \underline{p}}$$

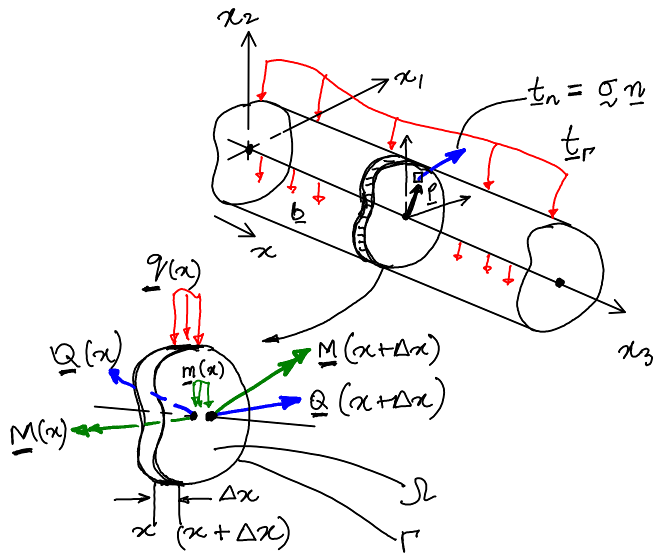
$$\boxed{\underline{u}(x) = \underline{\omega}(x) + \underline{\theta} \times \underline{p}}$$

$$u_1(x_1, x_2, x_3) = \omega_1(x_3) - x_2\, \theta_3(x_3)$$

$$u_2(x_1 x_2 x_3) = \omega_2(x_3) + x_1\, \theta_3(x_3)$$

$$u_3(x_1 x_2 x_3) = \omega_3(x_3) - x_1 \theta_2(x_3) + x_2\, \theta_1(x_3)$$

Resultant "Section" Forces and Moments



- Equilibrium:
  - Internal Section Forces (Shear + Axial)

$$\boxed{\underset{\sim}{Q}(x) \equiv \int_{\Omega} \underset{\sim}{t_n}\, dA}$$

  - Internal section moment: (Bending + Torsion)

$$\boxed{\underset{\sim}{M}(x) \equiv \int_{\Omega} \underset{\sim}{\rho} \times \underset{\sim}{t_n}\, dA}$$

  - Applied Section Forces:

$$\underset{\sim}{q}(x) \equiv \int_{\Omega} \underset{\sim}{b}\, d\Omega + \int_{\Gamma} \underset{\sim}{t_\Gamma}\, d\Gamma$$

  - Applied section moments:

$$\underset{\sim}{m}(x) \equiv \int_{\Omega} \underset{\sim}{\rho} \times \underset{\sim}{b}\, d\Omega + \int_{\Gamma} \underset{\sim}{\rho} \times \underset{\sim}{t_\Gamma}\, d\Gamma$$

$\Sigma \underset{\sim}{F} = 0$ @ every point x along the beam

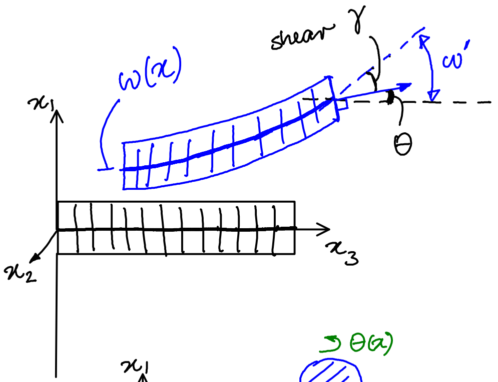
$$\Rightarrow \boxed{\frac{d\underset{\sim}{Q}}{dx} + \underset{\sim}{q} = \underset{\sim}{0}} \qquad (0 < x < L)$$

$\Sigma \underset{\sim}{M} = 0$ @ every point x along the beam

$$\Rightarrow \boxed{\frac{d\underset{\sim}{M}}{dx} + \left(\underset{\sim}{e_3} \times \underset{\sim}{Q}\right) + \underset{\sim}{m} = \underset{\sim}{0}} \qquad (0 < x < L)$$

Planar (2D) Beam Formulation

In the $x_1 - x_3$ plane:

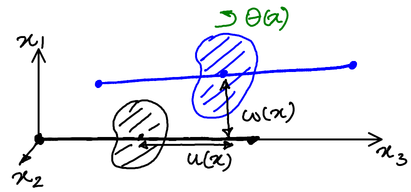


Generalized "Displacements"

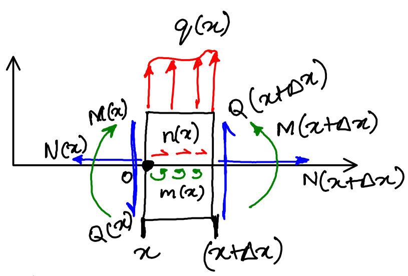
$$u(x) \equiv w_3(x_3) \qquad \text{Axial}$$
$$w(x) \equiv w_1(x_3) \qquad \text{Transverse}$$
$$\theta(x) \equiv \theta_2(x_3) \qquad \text{Rotation}$$

Generalized "Forces"

$$N(x) \equiv Q_3(x_3) \qquad \text{Axial force}$$
$$Q(x) \equiv Q_1(x_3) \qquad \text{shear force}$$
$$M(x) \equiv M_2(x_3) \qquad \text{Bending moment}$$



$n(x)$: Applied axial force per length
$q(x)$: Applied transverse load per length
$m(x)$: Applied moment per length



Governing Equations of Equilibrium for Beams

$$\Sigma F_3 = 0 \Rightarrow \quad N(x+\Delta x) - N(x) + n(x)\,\Delta x = 0 \quad \Rightarrow \quad \frac{dN}{dx} + n = 0$$

$$\Sigma F_1 = 0 \Rightarrow \quad Q(x+\Delta x) - Q(x) + q(x)\,\Delta x = 0 \quad \Rightarrow \quad \frac{dQ}{\partial x} + q = 0$$

$$\Sigma M_{2_0} = 0 \Rightarrow \quad M(x+\Delta x) - M(x) + Q(x+\Delta x).\Delta x + m(x)\,\Delta x + \left(q(x)\,\Delta x\right)\frac{\Delta x}{2} = 0$$

$$\Rightarrow \quad \frac{dM}{dx} + Q + m = 0$$

i.e.

$$\left.\begin{array}{rl} \text{Axial:} & \boxed{N' + n = 0} \\ \text{Transverse:} & \boxed{Q' + q = 0} \\ \text{Bending Moment:} & \boxed{M' + Q + m = 0} \end{array}\right\} \text{for all } x \in (0, L)$$

Using the Kinematic Assumption, we can find:

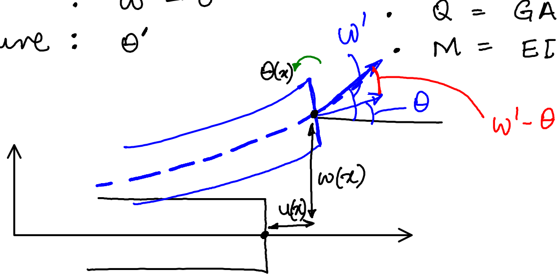
Strains:

- Axial : u'
- Shear : $w' - \theta$
- Curvature : θ'

Constitutive Relations

- $N = EA\,(u')$
- $Q = GA\,(w' - \theta)$
- $M = EI\,(\theta')$

$(G = \mu)$



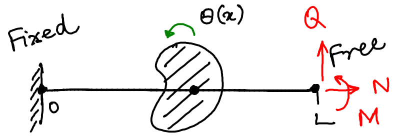
Governing Equations for "Timoshenko" Beam: (including shear)

Axial: $\quad (EAu')' + n = 0$

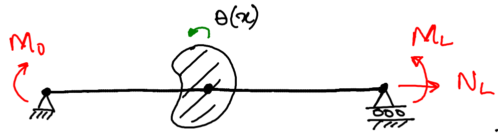
Transverse: $\quad (GA(w'-\theta))' + q = 0$

Bending: $\quad (EI(\theta'))' + GA(w'-\theta) + m = 0$

Boundary conditions:



EBC $\left[\begin{array}{l} \text{Axial} : u(0) = 0 \\ \text{Transverse} : w(0) = 0 \\ \text{Bending} : \theta(0) = 0 \end{array}\right.$ $\qquad$ $\left.\begin{array}{l} N(L) = (EAu') = \text{specified} \\ Q(L) = GA(w'-\theta) = \text{''} \\ M(L) = EI(\theta') = \text{''} \end{array}\right]$ NBC



EBC $\left[\begin{array}{l} \text{Axial} : \\ \text{Transverse} : \end{array}\right.$ $\quad \begin{array}{l} u(0) = 0 \\ w(0) = 0 \end{array}$ $\quad \begin{array}{l} N(L) = N_L \,] \text{ NBC} \\ w(L) = 0 \,] \text{ EBC} \end{array}$

NBC [Bending : $\quad M(0) = M_0 \quad M(L) = M_L$] NBC

Weak Form:

$$G(\underbrace{\{u, w, \theta\}}_{\underline{U}}, \underbrace{\{\bar{u}, \bar{w}, \bar{\theta}\}}_{\underline{\bar{U}}}) \equiv \int_0^L \bar{u}\left((EAu')' + n\right) dl$$

$$+ \int_0^L \bar{w}\left([GA(w'-\theta)]' + q\right) dl$$

$$+ \int_0^L \bar{\theta}\left((EI\,\theta')' + GA(w'-\theta) + m\right) dl$$

Integrating by parts:-

$$G(\underline{U}, \underline{\bar{U}}) = -\left[\int_0^L (\bar{u}'\, EAu' - \bar{u}\, n)\, dl\right] + \left[\bar{u}(EAu')\right]_0^L$$

$$- \left[\int_0^L \left(\bar{w}'\, GA(w'-\theta) - \bar{w}\, q\right) dl\right] + \left[\bar{w}\left(GA(w'-\theta)\right)\right]_0^L$$

$$- \left[\int_0^L \left(\bar{\theta}'\, EI\theta' - \bar{\theta}\, GA(w'-\theta) - \bar{\theta}\, m\right) dl\right] + \left[\bar{\theta}(EI\theta')\right]_0^L$$

Rearranging

$$\boxed{\begin{aligned} G(\underline{U},\underline{\bar{U}}) = &-\int_0^L \left[\bar{u}' EA u' + (\bar{\omega}'-\bar{\theta})\, GA(\omega'-\theta) + \bar{\theta}' EI \theta'\right] dl \\ &+ \int_0^L \left[\bar{u} n + \bar{\omega} q + \bar{\theta} m\right] dl + \left[\bar{u} N\right]_0^L + \left[\bar{\omega} Q\right]_0^L + \left[\bar{\theta} M\right]_0^L \end{aligned}}$$

Requirements for continuity:

To Solve $G(\underline{U},\underline{\bar{U}}) = 0$ for all $\underline{\bar{U}} = \{\bar{u},\bar{\omega},\bar{\theta}\} \in H_0^1(0,L)$

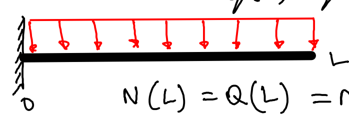
This means that we can choose our previous 1-D functions for $\{\bar{u},\bar{\omega},\bar{\theta}\}$ and the same framework will apply here.

For the Ritz method:

$$\left.\begin{aligned} u(x) &\approx u^h(x) = c_0 + \sum c_i f_i(x) \\ \omega(x) &\approx \omega^h(x) = a_0 + \sum a_i h_i(x) \\ \theta(x) &\approx \theta^h(x) = b_0 + \sum b_i g_i(x) \end{aligned}\right\} \quad \text{where } f_i, g_i, h_i \in H_0^1(0,L)$$

Example: Cantilever Beam:

$q(x) = q_0$



BC:

$u(0) = \omega(0) = \theta(0) = 0$ $\quad N(L) = Q(L) = M(L) = 0$

Assume Polynomial approximation:
(axial approximation eliminated)

$$w(x) = a_1 x + a_2 \frac{x^2}{\ell}, \quad \theta(x) = b_1 \frac{x}{\ell}$$

$$\overline{w}(x) = \bar{a}_1 x + \bar{a}_2 \frac{x^2}{\ell}, \quad \theta(x) = \bar{b}_1 \frac{x}{\ell}$$

Weak form:

$$G(\underline{U},\underline{\bar{U}}) = \{\bar{a}_1 \;\; \bar{a}_2 \;\; \bar{b}_1\} \left\{ \frac{GA\ell}{12} \begin{bmatrix} 12 & 12 & -6 \\ 12 & 16 & -8 \\ -6 & -8 & 4+\beta \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ b_1 \end{bmatrix} - \left(-\frac{q_0 \ell^2}{6}\right) \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \right\} = 0$$

Solving the equations:

$$a_1 = -\frac{q\ell}{GA}, \quad a_2 = \frac{q\ell}{GA}\left(\frac{\beta-2}{2\beta}\right), \quad b_1 = -\frac{q\ell}{GA}\left(\frac{2}{\beta}\right)$$

where $\beta = \frac{12}{\ell^2} \frac{EI}{GA}$

Thus:

$$w(x) = \frac{q\ell^4}{24EI}\left((\beta-2)\frac{x^2}{\ell^2} - 2\beta \frac{x}{\ell}\right), \quad \theta(x) = -\frac{q\ell^3}{6EI}\left(\frac{x}{\ell}\right)$$

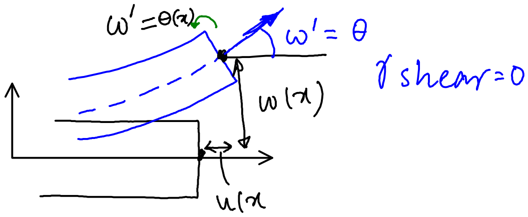
Note: We have used Quadratic Polynomial for $\omega(x)$ and a Linear Polynomial for $\theta(x)$

In general, we choose approximation of $\omega(x)$ one degree higher than $\theta(x)$.

Governing Equations for a Bernoulli-Euler Beam: (neglecting shear)

If we further assume that the shear deformation is zero:

$$w' - \theta = 0 \quad \Rightarrow \quad \boxed{w' = \theta}$$



(i.e. the beam is rigid in shear),

Then equilibrium:-

$$\text{Axial} : \quad (EAu')' + n = 0$$

$$\text{Transverse \& Bending} : \quad (EI\,w'')' + Q + m = 0$$

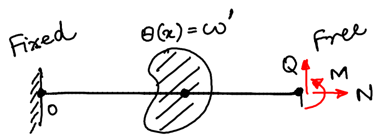
$$\text{i.e.} \quad \boxed{(EI\,w'')'' = q - m'} \qquad \text{(Differentiate again)}$$

Note:
- We have eliminated another unknown: $\theta(x) = w'(x)$
- We also lost the constitutive equation $Q = GA(w' - \theta)$.

  Shear must now be calculated from "Statics" (equilibrium):

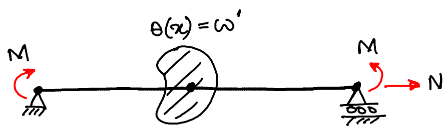
  $$Q = -M' - m = -(EI\,w'')' - m$$

Boundary conditions:



EBC [Axial : $u(0) = 0$; Transverse : $w(0) = 0$; Bending : $w'(0) = 0$]

[$N(L) = (EAu') =$ specified; $Q(L) = -(EI\,w'')' - m =$ ″; $M(L) = EI\,\theta' = EI\,w'' =$ ″] NBC



EBC	Axial : $u(0) = 0$	$N(L) = $ spec	NBC
EBC	Transverse : $w(0) = 0$	$w(L) = 0$	EBC
NBC	Bending : $EI\,w''(0) =$ specified	$EI\,w''(L) =$ spec	NBC

Weak Form:

$$G(\{u, w\}, \{\bar{u}, \bar{w}\}) \equiv \int_0^L \bar{u}\left((EA\,u')' + n\right) dl$$

$$+ \int_0^L \bar{w}\left((EI\,w'')'' - q + m'\right) dl$$

Weak Form for Bernoulli-Euler beams: (Bending Only)

$$G(w,\bar{w}) \equiv \int_0^L \bar{w}\left((EIw'')'' - q + m'\right) dl$$

Integrate by parts: (twice)

$$G(w,\bar{w}) = -\int_0^L \left[\bar{w}'(EIw'')' + \bar{w}(q-m')\right] dl + \left[\bar{w}(EIw'')'\right]_0^L$$

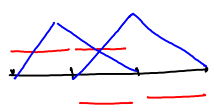
$$\boxed{G(w,\bar{w}) = \int_0^L \left[\bar{w}''(EIw'') - \bar{w}(q-m')\right] dl + \left[\bar{w}(EIw'')'\right]_0^L - \left[\bar{w}'(EIw'')\right]_0^L}$$

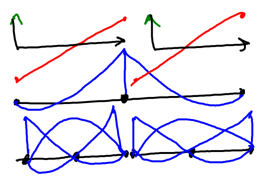
Note: • Continuity requirements:

$$G(w,\bar{w}) = 0 \quad \forall \quad \bar{w} \in H_0^2(0,L)$$

• The chosen functions must also be complete atleast upto polynomial order 2 (because derivatives of order 2 : w'')

• Conventional 1-D functions - not ok.

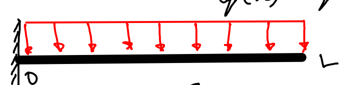




Ritz Method

$$w(x) \approx w^h(x) = a_0 + \sum_{i=1}^{n} a_i h_i(x) \qquad h_i(x) \in H_0^2(0,L)$$

Example Cantilever Beam:



BC:

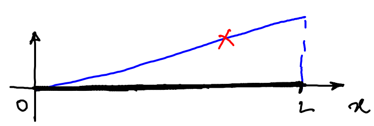
$$\text{EBC}\begin{cases} w(0) = 0 \\ w'(0) = 0 \end{cases} \qquad \text{NBC}\begin{cases} Q(L) = -EIw''' = 0 \\ M(L) = EIw'' = 0 \end{cases}$$

$a_0 = 0$;

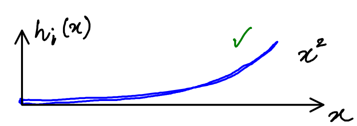
$h_1(x) = x^2$

$w(x) = a_1 x^2 \quad \Rightarrow \quad w''(x) = 2a_1$

$\bar{w}(x) = \bar{a}_1 x^2 \qquad \bar{w}''(x) = 2\bar{a}_1$



$$G(w,\bar{w}) = \int_0^L \bar{a}_1 (EI \times 2 \times 2)\, a_1 dl - \int_0^L \bar{a}_1 x^2 q_0\, dl$$



$$G(w,\bar{w}) = \bar{a}_1\left(\underbrace{[4EIL]}_{K}\underbrace{\{a_1\}}_{d} - \underbrace{\left\{\frac{L^3}{3}q_0\right\}}_{f}\right) = 0 \qquad \forall\ \bar{a}_1$$

$$\Rightarrow \underline{a_1 = \frac{q_0 L^2}{12EI}} \qquad \Rightarrow \boxed{w(x) = a_1 x^2 = \frac{q_0 L^2 x^2}{12EI}}$$

Exact Solution: $\quad (EI\,w'')'' = q_0$

Integrate: $\Rightarrow \quad (EI\,w'')' = q_0 x + C_1$

BC : $Q(L) = 0 \Rightarrow -(EI\,w'')' = -(q_0 L + C_1) = 0 \Rightarrow \quad C_1 = -q_0 L$

Integrate $\Rightarrow \quad (EI\,w'') = q_0 \frac{x^2}{2} - q_0 L x + C_2$

BC: $M(L) = 0 \Rightarrow (EI w'') = \frac{q_0 L^2}{2} - q_0 L^2 + C_2 = 0 \Rightarrow C_2 = \frac{q_0 L^2}{2}$

Integrate $\Rightarrow \quad EI w' = \frac{q_0 x^3}{6} - \frac{q_0 L x^2}{2} + \frac{q_0 L^2}{2} x + C_3$

BC $\quad w'(0) = 0 \quad \Rightarrow \quad C_3 = 0$

Integrate $\Rightarrow \quad w(x) = \frac{1}{EI}\left(\frac{q_0 x^4}{24} - \frac{q_0 L x^3}{6} + \frac{q_0 L^2}{4} x^2 + C_4\right)$

BC $\quad w(0) = 0 \quad \Rightarrow \quad C_4 = 0$

Thus $\quad w(x) = \frac{1}{EI}\left(\frac{q_0 x^4}{24} - \frac{q_0 L x^3}{6} + \frac{q_0 L^2 x^2}{4}\right)$

Comparing the Ritz solution @ $x = L$:

Ritz : $\frac{q_0 L^4}{12 EI}$; Exact : $\frac{q_0 L^4}{8 EI}$

Variational Principles for Beams

$$\begin{tikzcd}
\Pi(\underline{U}) \arrow[r, bend right, "D\Pi(u)\cdot\bar{u}"'] & G(\underline{U},\underline{\bar{U}}) \arrow[l, bend right] \arrow[r, bend right, "\text{Euler Eq.}"'] & \left[\begin{array}{c} N'+n=0 \\ Q'+q=0 \\ M'+Q+m=0 \end{array}\right] \arrow[l, bend right, "\text{MWR}"']
\end{tikzcd}$$

(i) Timoshenko (shear) beam:

$$\Pi(\underbrace{\{u, w, \theta\}}_{\underline{U}}) = \int_0^L \left[\frac{1}{2} EA(u')^2 + \frac{1}{2} GA(w'-\theta)^2 + \frac{1}{2} EI(\theta')^2\right] dl$$
$$- \int_0^L [u.n + w.q + \theta.m]\, dl$$
$$- [u N]_0^L - [w Q]_0^L - [\theta M]_0^L$$

Check Minimizing $\Pi(\underline{U})$:

$$D\Pi(\underline{U})\cdot\underline{\bar{U}} = \left[\frac{d}{d\epsilon} \Pi(\underline{U}+\epsilon\underline{\bar{U}})\right]\Bigg|_{\epsilon=0}$$

$$= \int_0^L \Big[EA\left(\bar{u}'u' + \epsilon\bar{u}'^2\right) + GA\left[(\bar{w}'-\bar{\theta})(w'-\theta) + \epsilon(\bar{w}'-\bar{\theta})^2\right]$$
$$+ EI\left(\bar{\theta}'\theta' + \epsilon\bar{\theta}'^2\right)\Big] dl - \int_0^L [\bar{u}.n + \bar{w}.q + \bar{\theta}.m]\, dl$$
$$- [\bar{u} N]_0^L - [\bar{w} Q]_0^L - [\bar{\theta} M]_0^L$$

Thus

$$D\Pi(\underline{U})\cdot\underline{\bar{U}} = G(\underline{U},\underline{\bar{U}})$$

(ii) Bernoulli-Euler Beam: (Bending only)

$$\Pi(w) = \int_0^L \frac{1}{2} EI(w'')^2\, dx + \int_0^L (w q - w m')\, dl - [w Q]_0^L - [w' M]_0^L$$

Check Minimizing:

$$D\Pi(w)\cdot\bar{w} = \left[\frac{d}{d\epsilon} \Pi(w + \epsilon\bar{w})\right]\Bigg|_{\epsilon=0}$$

$$= \int_0^L EI\,\bar{w}''w''\, dx + \int_0^L (\bar{w} q - \bar{w} m')\, dx - [\bar{w} Q]_0^L - [\bar{w}' M]_0^L$$

$$= G(w,\bar{w})$$

Finite Element Approximations for Beams

(i) Timoshenko Beams (with shear)

Weak form:

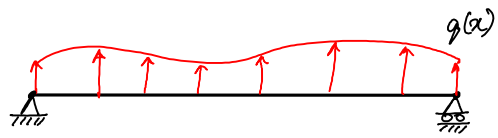
$$\boxed{\begin{aligned} G(\underline{U},\underline{\bar{U}}) &= -\int_0^L \left[\bar{u}' EA u' + (\bar{w}'-\bar{\theta}) GA (w'-\theta) + \bar{\theta}' EI \theta'\right] dl \\ &\quad + \int_0^L \left[\bar{u} n + \bar{w} q + \bar{\theta} m\right] dl + \left[\bar{u} N\right]_0^L + \left[\bar{w} Q\right]_0^L + \left[\bar{\theta} M\right]_0^L \end{aligned}}$$

Highest degree of derivatives = 1

$\Rightarrow \{\bar{u}, \bar{w}, \bar{\theta}\}$ must be in $H_0^1(0,l)$

Thus we can use our 1-D finite elements for Timoshenko beams:

<u>Example:</u> Simply supported beam:

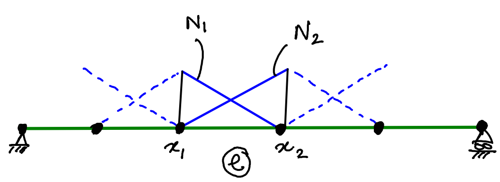


Ritz Approximation:

$$\left.\begin{aligned} u(x) &\approx u^h(x) = c_0 + \sum c_i f_i(x) \\ w(x) &\approx w^h(x) = a_0 + \sum a_i h_i(x) \\ \theta(x) &\approx \theta^h(x) = b_0 + \sum b_i g_i(x) \end{aligned}\right\}$$

where $f_i, g_i, h_i \in H_0^1(0,L)$

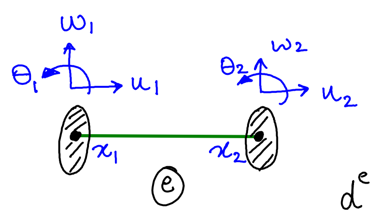
Equivalent Finite element approximation:
for element (e):



$$u(x) = [N_1 \quad N_2]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$w(x) = [N_1 \quad N_2]\begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix}$$

$$\theta(x) = [N_1 \quad N_2]\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}$$



Element degrees of freedom

$$\underline{d}^e \equiv \{\underline{u_1\ u_2}\ \underline{w_1\ w_2}\ \underline{\theta_1\ \theta_2}\}$$

Substituting these expressions into the weak form:

$$\tilde{G}^h(\underline{d}, \underline{\bar{d}}) = \underline{\bar{d}}^{g^T}\left[\underset{\sim}{K}^g \underline{d}^g - \underline{f}^g\right] = 0 \qquad \text{for all } \underline{\bar{d}}^g$$

$$[\underset{\sim}{K}^g] = \underset{e=1}{\overset{M}{\mathsf{A}}}[\underset{\sim}{K}^{el}] \qquad ; \qquad \underline{f}^g = \underset{e=1}{\overset{M}{\mathsf{A}}} \underline{f}^{el}$$

Note:

$$\underset{6\times6}{[\underset{\sim}{K}^{el}]} = \left[\begin{array}{c|c|c} K_{uu} & 0 & 0 \\ \hline 0 & K_{ww} & K_{w\theta} \\ \hline 0 & K_{\theta w} & K_{\theta\theta} \end{array}\right]$$

where

$$K_{uu} = \int_{x_1^e}^{x_2^e} \begin{bmatrix} N_1' \\ N_2' \end{bmatrix} EA \begin{bmatrix} N_1' & N_2' \end{bmatrix} dx \Rightarrow K_{uu} = \frac{EA}{l_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$K_{ww} = \int_{x_1^e}^{x_2^e} \begin{bmatrix} N_1' \\ N_2' \end{bmatrix} GA \begin{bmatrix} N_1' & N_2' \end{bmatrix} dx \Rightarrow K_{ww} = \frac{GA}{l_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$K_{w\theta} = K_{\theta w}^T = \int_{x_1^e}^{x_1^e} \begin{bmatrix} N_1' \\ N_2' \end{bmatrix} GA \begin{bmatrix} N_1 & N_2 \end{bmatrix} dx = \begin{bmatrix} * & * \\ * & * \end{bmatrix}$$

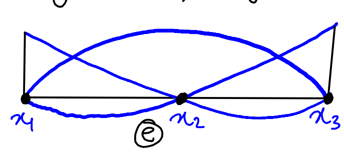
$$K_{\theta\theta} = \int_{x_1^e}^{x_2^e} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} GA \begin{bmatrix} N_1 & N_2 \end{bmatrix} dx + \int_{x_1^e}^{x_2^e} \begin{bmatrix} N_1' \\ N_2' \end{bmatrix} EI \begin{bmatrix} N_1' & N_2' \end{bmatrix} dx = \begin{bmatrix} * & * \\ * & * \end{bmatrix}$$

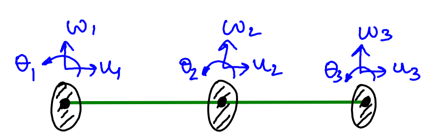
Element force vector:

$$\underline{f}^{el} = \int_{x_1^e}^{x_2^e} \left[\begin{array}{c|c|c} N_1 & & \\ N_2 & & \\ \hline & N_1 & \\ & N_2 & \\ \hline & & N_1 \\ & & N_2 \end{array}\right] \begin{Bmatrix} n \\ q \\ m \end{Bmatrix} dx + \left[\bar{u}\,N\right]_0^L + \left[\bar{w}\,Q\right]_0^L + \left[\bar{\theta}\,M\right]_0^L$$

Higher Order Lagrange shape functions can also be used

$$\underset{\sim}{N} = \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix}$$





Shear Locking in Timoshenko beams

Just like the Ritz method by Timoshenko beams, equal order interpolation for $w(x)$ and $\theta(x)$ leads to shear locking.

To remedy shear locking:

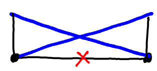
(i) Reduced Integration

→ Choose equal order interpolation for $w(x)$ and $\theta(x)$ and use "reduced" Gauss Points to integrate:

$$\int_0^l (\bar{w}' - \bar{\theta})\, GA (w' - \theta)\, dx \qquad \text{(shear term)}$$

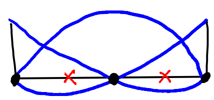
→ For example

- $w(x)$: linear ⇒ w' : constant
  $\theta(x)$: linear



⇒ Use 1 gauss point

- $w(x)$: quadratic ⇒ w' : linear
  $\theta(x)$: quadratic



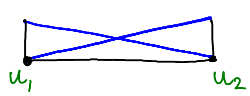
⇒ Use 2 gauss points

(ii) Consistent interpolation

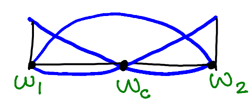
→ Choose shape functions such that polynomial order of $w'(x)$ and $\theta(x)$ is the same.

→ For example

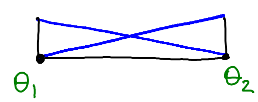
$u(x)$: linear



$w(x)$: quadratic



$\theta(x)$: linear



element dofs: $\underline{d} = \{ u_1\, u_2 \quad w_1\, w_c\, w_2 \quad \theta_1\, \theta_2 \}$

$$[K^{el}]_{7\times 7} = \begin{bmatrix} K_{uu} & & \\ & K_{ww} & K_{w\theta} \\ & K_{\theta w} & K_{\theta\theta} \end{bmatrix}$$

$[K_{uu}]_{2\times 2}$

$[K_{ww}]_{3\times 3}$ $\quad [K_{w\theta}]_{3\times 2}$

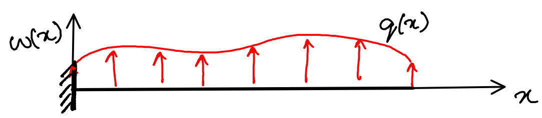
$[K_{\theta w}]_{2\times 3}$ $\quad [K_{\theta\theta}]_{2\times 2}$

(ii) Bernoulli-Euler Beams:

Recall weak form:

$$\boxed{G(w,\bar{w}) = \int_0^L \left[\bar{w}''(EI\,w'') - \bar{w}\,(q - m')\right]dl + \left[\bar{w}\,(EI\,w'')'\right]_0^L - \left[\bar{w}'(EI\,w'')\right]_0^L}$$

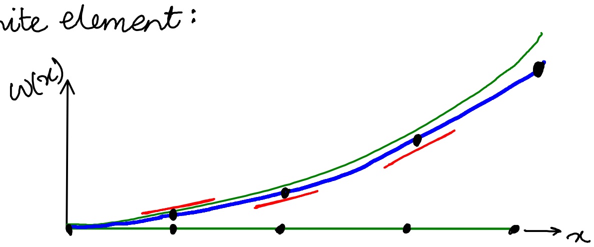
Consider a cantilever beam:



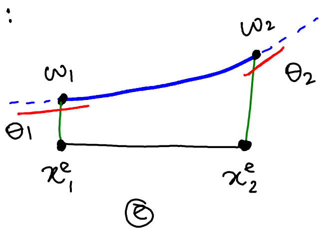
$$(EIw'')'' - q = 0$$

Weak form $\quad G(w,\bar{w}) = 0 \quad \forall\, \bar{w}$

Finite element:



One Element:



$$w(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

$$\theta(x) = w'(x) = a_1 + 2a_2 x + 3a_3 x^2$$

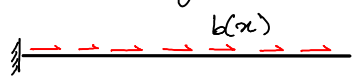
such that:

$$\left.\begin{aligned} w_1 &= a_0 + a_1 x_1^e + a_2 {x_1^e}^2 + a_3 {x_1^e}^3 \\ \theta_1 &= a_1 + 2a_2 x_1^e + 3a_3 {x_1^e}^2 \\ w_2 &= a_0 + a_1 x_2^e + a_2 {x_2^e}^2 + a_3 {x_2^e}^3 \\ \theta_2 &= a_1 + 2a_2 x_2^e + 3a_3 {x_2^e}^2 \end{aligned}\right\}$$

Element dofs

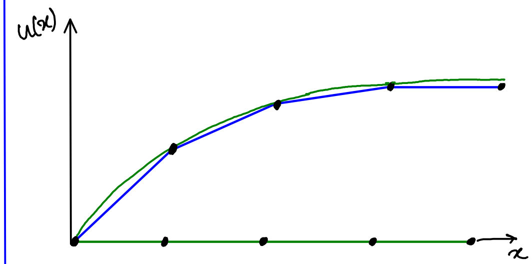
$$\begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} = \begin{bmatrix} 1 & x_1^e & {x_1^e}^2 & {x_1^e}^3 \\ 0 & 1 & 2x_1^e & 3{x_1^e}^2 \\ 1 & x_2^e & {x_2^e}^2 & {x_2^e}^3 \\ 0 & 1 & 2x_2^e & 3{x_2^e}^2 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

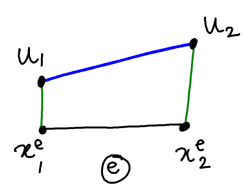
1-D analogous problem



$$(cu')' + b = 0$$

$$G(u,\bar{u}) = 0 \quad \forall\, \bar{u}$$





$$u(x) = a_0 + a_1 x$$

such that

$$w_1 = u(x_1^e) = a_0 + a_1 x_1^e$$

$$u_2 = u(x_2^e) = a_0 + a_1 x_2^e$$

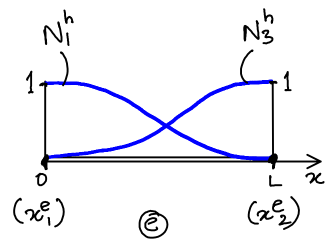
Element dofs

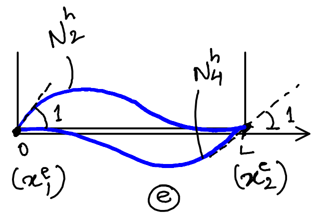
$$\Rightarrow \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} 1 & x_1^e \\ 1 & x_2^e \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix}$$

Thus

$$\begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1^e & {x_1^e}^2 & {x^e}^3 \\ 0 & 1 & 2x_1^e & 3{x_1^e}^2 \\ 1 & x_2^e & {x_2^e}^2 & {x_2^e}^3 \\ 0 & 1 & 2x_2^e & 3{x_2^e}^2 \end{bmatrix}^{-1} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix}$$

$$\Rightarrow w(x) = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix}$$





$$N_1^h(x) = 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3} \qquad N_2^h(x) = x - \frac{2x^2}{L} + \frac{x^3}{L^2}$$

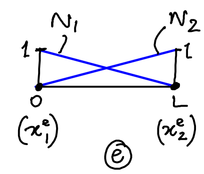
$$N_3^h(x) = \frac{3x^2}{L^2} - \frac{2x^3}{L^3} \qquad N_4^h(x) = -\frac{x^2}{L} + \frac{x^3}{L^2}$$

$$\Rightarrow \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} = \begin{bmatrix} 1 & x_1^e \\ 1 & x_2^e \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Substitute $\{a_0 \ a_1\}$ into $u(x) = a_0 + a_1 x$

$$\Rightarrow u(x) = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

i.e. $\underline{u} = \underset{\sim}{N} \, \underline{d}$



$$N_1(x) = 1 - \frac{x}{L}$$

$$N_2(x) = \frac{x}{L}$$

These are called Hermite functions. They impose the continuity of the primary variable $w(x)$ and its derivative $(w'(x) = \theta(x))$.

Substituting this Finite Element approximation into the weak form: (including axial)

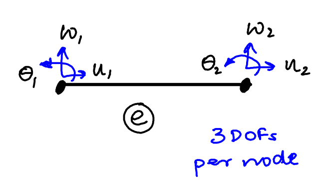
$$G(\{u, w\}, \{\bar{u}, \bar{w}\}) = \sum_{e=1}^{M} \left[\int_{x_1^e}^{x_2^e} \left[(\bar{u}' EA \, u') + (\bar{w}'' EI \, w'') \right] dx - \int_{x_1^e}^{x_2^e} (\bar{u} n + \bar{w} q - m) \, dx - [\bar{u} N]_0^L - [\bar{w} Q]_0^L - [\bar{w}' M]_0^L \right]$$

$$u(x) = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$w(x) = \begin{bmatrix} N_1^h & N_2^h & N_3^h & N_4^h \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix}$$

Element Dofs: $\{\underline{u_1 u_2} \ \underline{w_1 \theta_1 \ w_2 \theta_2}\}$

$$\tilde{G}^h(d, \bar{d}) = \mathop{\mathrm{A}}_{e=1}^{M} \underline{\bar{d}}^{g^T} (\underset{\sim}{K}^g \underline{d}^g - \underline{f}^g) = 0 \quad \forall \ \underline{\bar{d}}$$



where $\underset{\sim}{K}^g = \underset{\sim}{K}^{el}$; $\underline{f}^g = \mathop{\mathrm{A}}_{e=1}^{M} \underline{f}^{el}$

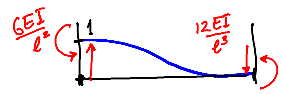
Note :

$$\underset{\sim}{K}^{el} = \left[\begin{array}{c|c} K_{uu} & \\ \hline & K_{ww} \end{array}\right]$$

$$\underset{\sim}{K}_{uu}^{el} = \int_{x_1^e}^{x_2^e} \begin{bmatrix} N_1' \\ N_2' \end{bmatrix} EA \begin{bmatrix} N_1' & N_2' \end{bmatrix} dx = \frac{EA}{\ell} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\ell = |x_2^e - x_1^e|$$

$$\underset{\sim}{K}_{ww}^{el} = \int_{x_1^e}^{x_2^e} \begin{bmatrix} N_1'' \\ N_2'' \\ N_3'' \\ N_4'' \end{bmatrix} EI \begin{bmatrix} N_1'' & N_2'' & N_3'' & N_4'' \end{bmatrix} dx = \frac{EI}{\ell^3} \begin{bmatrix} \boxed{\begin{matrix} 12 \\ 6\ell \\ -12 \\ 6\ell \end{matrix}} & \begin{matrix} 6\ell \\ 4\ell^2 \\ -6\ell \\ 2\ell^2 \end{matrix} & \begin{matrix} -12 \\ -6\ell \\ 12 \\ -6\ell \end{matrix} & \begin{matrix} 6\ell \\ 2\ell^2 \\ -6\ell \\ 4\ell^2 \end{matrix} \end{bmatrix}$$

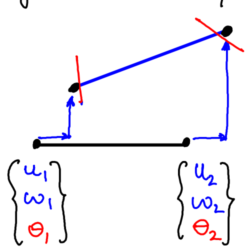


Element force vector :

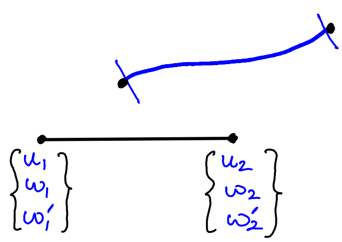
$$\underline{f}^{el} = \int_{x_1^e}^{x_2^e} \left[\begin{array}{c|c} N_1 & \\ N_2 & \\ \hline & N_1^h \\ & N_2^h \\ & N_3^h \\ & N_4^h \end{array}\right] \begin{Bmatrix} n \\ q - m' \end{Bmatrix} dx + \left[\bar{u}\, N\right]_0^L + \left[\bar{w}\, Q\right]_0^L + \left[\bar{w}' M\right]_0^L$$

Post processing : After solving $\underset{\sim}{K}^g \, \underline{d}^g = \underline{f}^g$

- Plot deformed shape



Timoshenko Beam with
Linear elements



Bernoulli-Euler beam with
Hermite-cubic elements

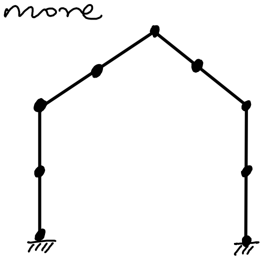
- Calculate

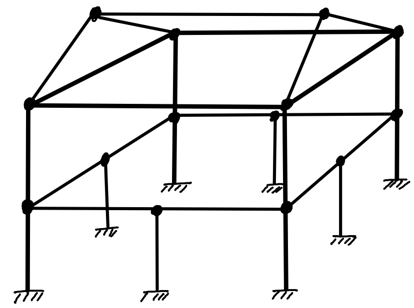
→ Axial force	$N(x)$	:	$EA\, u'(x)$	:	$EA\, u'$
→ Shear force	$Q(x)$	:	$GA\,(w' - \theta)$	:	$Q = -M' - m = -EI w'''$
→ Bending Moment	$M(x)$	:	$EI\,(\theta')$	:	$M = EI w''$
			(Timoshenko)		(Bernoulli-Euler)

Frame Structures in 2D & 3D

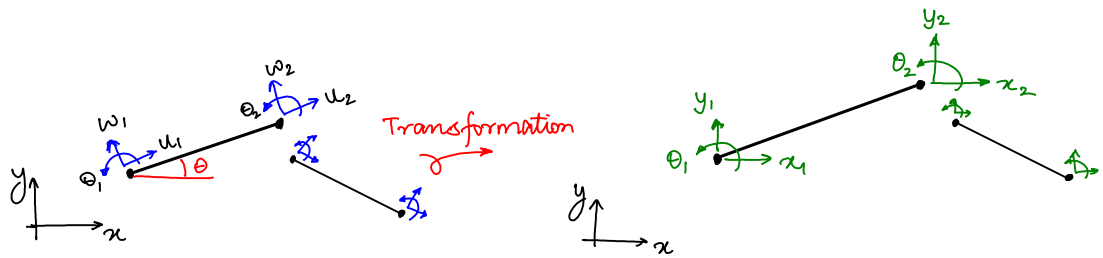
Each beam / column member can be modelled with one or more beam elements.

(Timoshenko / Bernoulli-Euler)





2D Frame Element



Element dofs in $\boxed{\text{local}}$ coordinates

$d_e^L = \{u_1\ w_1\ \theta_1\ \ u_2\ w_2\ \theta_2\}$

Cannot be assembled.

Element dofs in $\boxed{\text{Global}}$ coordinates

$d_e^G \{x_1\ y_1\ \theta_1\ \ x_2\ y_2\ \theta_2\}$

Can be assembled.

i.e

$$\underbrace{\left[\begin{array}{ccc|ccc} \cos\theta & -\sin\theta & 0 & & & \\ \sin\theta & \cos\theta & 0 & & 0 & \\ 0 & 0 & 1 & & & \\ \hline & & & \cos\theta & -\sin\theta & 0 \\ & 0 & & \sin\theta & \cos\theta & 0 \\ & & & 0 & 0 & 1 \end{array}\right]}_{\underset{\sim}{T}} \underbrace{\left\{\begin{array}{c} u_1 \\ w_1 \\ \theta_1 \\ \hline u_2 \\ w_2 \\ \theta_2 \end{array}\right\}}_{d_e^L} = \underbrace{\left\{\begin{array}{c} x_1 \\ y_1 \\ \theta_1 \\ \hline x_2 \\ y_2 \\ \theta_2 \end{array}\right\}}_{d_e^G}$$

$$\Rightarrow \quad \underset{\sim}{d_e^L} = \underset{\sim}{T}^T \underset{\sim}{d_e^G}$$

The element stiffness matrix and load vectors can also be Transformed:

$$\tilde{G}^h(\underset{\sim}{d}, \underset{\sim}{\bar{d}}) = -\sum_{e=1}^{M} \bar{d}_e^{L^T}\left(\underset{\sim}{K_e^L}\, d_e^L - \underset{\sim}{f_e^L}\right)$$

$$= -\sum_{e=1}^{M} \left(\underset{\sim}{T}^T \bar{d}_e^G\right)^T \left[\underset{\sim}{K_e^L}\left(\underset{\sim}{T}^T d_e^G\right) - \left(\underset{\sim}{T}^T f_e^G\right)\right]$$

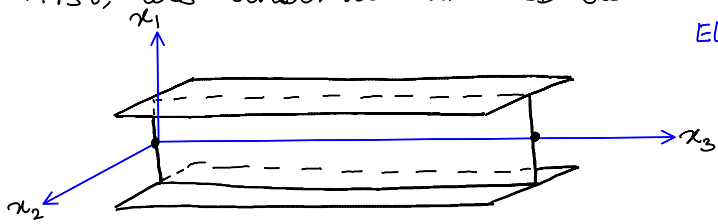
$$= -\sum_{e=1}^{M} \bar{d}_e^{G^T}\left[\underbrace{\left(\underset{\sim}{T}\, \underset{\sim}{K_e^L}\, \underset{\sim}{T}^T\right)}_{} d_e^G - f_e^G\right]$$

$$= -\underset{e=1}{\overset{M}{\mathsf{A}}} \bar{d}_e^{G^T}\left[\underset{\sim}{K_e^G}\, d_e^G - f_e^G\right]$$

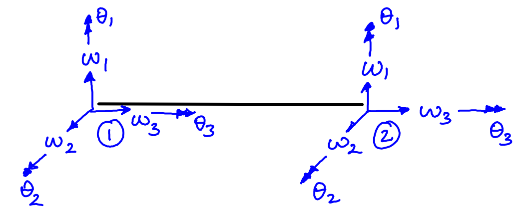
$$\Rightarrow \quad -\bar{d}^{G^T}\left[\underset{\sim}{K}^G d^G - f^G\right] = 0 \qquad \forall \bar{d}^G$$

3D Frame Structures

First, lets construct the 3D Element in local coordinates:



Element dofs in Local Coordinates:



For Bernoulli-Euler beam

6 Dofs per node

Bending in $x_1 x_3$ plane:

$$\underset{\sim}{K}^e_y = \frac{EI_y}{\ell^3}\begin{bmatrix} 12 & 6\ell & -12 & 6\ell \\ 6\ell & 4\ell^2 & -6\ell & 2\ell^2 \\ -12 & -6\ell & 12 & -6\ell \\ 6\ell & 2\ell^2 & -6\ell & 4\ell^2 \end{bmatrix} \begin{array}{l} w_1 \\ \theta_2 \\ w_1 \\ \theta_2 \end{array} \begin{array}{c} (1) \\ \\ (2) \\ \end{array}$$

Axial x_3:

$$\underset{\sim}{K}^e = \frac{EA}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Torsion in x_3:

$$\underset{\sim}{K}^e_z = \frac{GJ}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$J = I_x + I_y$$

(Polar moment of inertia)

Bending in $x_2 x_3$ plane:

$$\underset{\sim}{K}^e_x = \frac{EI_x}{\ell^3}\begin{bmatrix} 12 & -6\ell & -12 & -6\ell \\ -6\ell & 4\ell^2 & 6\ell & 2\ell^2 \\ -12 & 6\ell & 12 & 6\ell \\ -6\ell & 2\ell^2 & 6\ell & 4\ell^2 \end{bmatrix} \begin{array}{l} w_2 \\ \theta_1 \\ w_2 \\ \theta_1 \end{array} \begin{array}{c} (1) \\ \\ (2) \\ \end{array}$$

Thus the element contribution:

$$\tilde{G}^h(\underset{\sim}{d}, \underset{\sim}{\bar{d}}) = -\sum_{e=1}^{M} \underset{\sim}{\bar{d}}{}_e^{L^T}\left(\underset{\sim}{K}{}_e^L\, \underset{\sim}{d}{}_e^L - \underset{\sim}{f}{}_e^L\right)$$

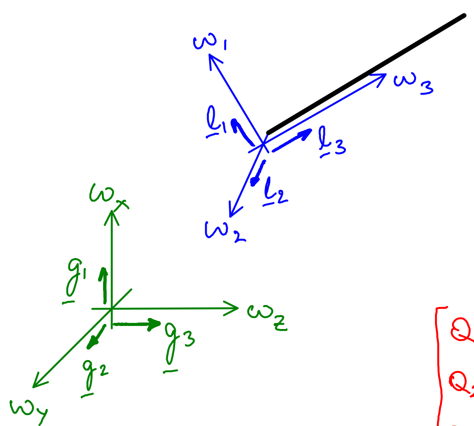
where

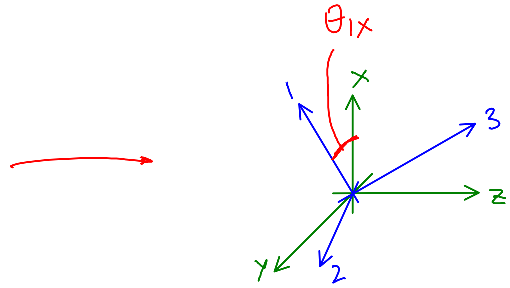
$$\left\{ w_{11}\; w_{12}\; w_{13}\; \theta_{11}\; \theta_{12}\; \theta_{13}\; w_{21}\; w_{22}\; w_{23}\; \theta_{21}\; \theta_{22}\; \theta_{23} \right\}$$

$$[\underset{\sim}{K}{}_e^L]\{\underset{\sim}{d}{}_e^L\} = \frac{1}{\ell^3}\left[\begin{array}{cccccc|cccccc} 12EI_y & & & & 6\ell EI_y & & -12EI_y & & & & 6\ell EI_y & \\ & 12EI_x & & -6\ell EI_x & & & & -12EI_x & & -6\ell EI_x & & \\ & & EA\ell^2 & & & & & & -EA\ell^2 & & & \\ & -6\ell EI_x & & 4\ell^2 EI_x & & & & 6\ell EI_x & & 2\ell^2 EI_x & & \\ 6\ell EI_y & & & & 4\ell^2 EI_y & & -6\ell EI_y & & & & 2\ell^2 EI_y & \\ & & & & & GJ\ell^2 & & & & & & -GJ\ell^2 \\ \hline -12EI_y & & & & -6\ell EI_y & & 12EI_y & & & & -6\ell EI_y & \\ & -12EI_x & & 6\ell EI_x & & & & 12EI_x & & 6\ell EI_x & & \\ & & -EA\ell^2 & & & & & & EA\ell^2 & & & \\ & -6\ell EI_x & & 2\ell^2 EI_x & & & & 6\ell EI_x & & 4\ell^2 EI_x & & \\ 6\ell EI_y & & & & 2\ell^2 EI_y & & -6\ell EI_y & & & & 4\ell^2 EI_y & \\ & & & & & -GJ\ell^2 & & & & & & GJ\ell^2 \end{array}\right] \left\{\begin{array}{c} w_{11} \\ w_{12} \\ w_{13} \\ \theta_{11} \\ \theta_{12} \\ \theta_{13} \\ \hline w_{21} \\ w_{22} \\ w_{23} \\ \theta_{21} \\ \theta_{22} \\ \theta_{23} \end{array}\right\} \begin{array}{c} \\ \\ (1) \\ \\ \\ \\ \\ \\ (2) \\ \\ \\ \\ \end{array}$$

3D Transformation

To be able to assemble the element matrices and vectors, we need to transform the local co-ordinates to global in 3D:





$$\underbrace{\begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix}}_{\underset{\sim}{Q}} \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} = \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix}$$

$Q_{11} = \cos\theta_{1x}$

where $Q_{ij} = \underline{g}_i \cdot \underline{\ell}_j$

Thus to transform the local dofs to global dofs:

$$\underbrace{\begin{bmatrix} \underset{\sim}{Q} & & & \\ & \underset{\sim}{Q} & & \\ & & \underset{\sim}{Q} & \\ & & & \underset{\sim}{Q} \end{bmatrix}}_{\underset{\sim}{T}} \underbrace{\begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \hline \omega_1 \\ \omega_2 \\ \omega_3 \\ \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix}}_{\underline{d}_e^L} \begin{matrix} (1) \\ \\ (2) \end{matrix} = \underbrace{\begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \\ \theta_x \\ \theta_y \\ \theta_z \\ \hline \omega_x \\ \omega_y \\ \omega_z \\ \theta_x \\ \theta_y \\ \theta_z \end{Bmatrix}}_{\underline{d}_e^G} \begin{matrix} (1) \\ \\ (2) \end{matrix}$$

Thus the weak form:

$$\tilde{G}^h(\underline{d}, \bar{\underline{d}}) = -\sum_{e=1}^{M} \bar{\underline{d}}_e^{L^T}\left(\underset{\sim}{K}_e^L \underline{d}_e^L - \underline{f}_e^L\right)$$

$$= -\sum_{e=1}^{M} \bar{\underline{d}}_e^{G^T}\left[\left(\underset{\sim}{T}\,\underset{\sim}{K}_e^L\,\underset{\sim}{T}^T\right)\underline{d}_e^G - \underline{f}_e^G\right]$$

$$= -\mathop{\large\mathsf{A}}_{e=1}^{M} \bar{\underline{d}}_e^{G^T}\left[\underset{\sim}{K}_e^G \underline{d}_e^G - \underline{f}_e^G\right]$$

Finally solve:

$$\Rightarrow \quad -\bar{\underline{d}}^{G^T}\left[\underset{\sim}{K}^G \underline{d}^G - \underline{f}^G\right] = 0 \quad \forall \bar{\underline{d}}^G \qquad \text{(with BCs)}$$