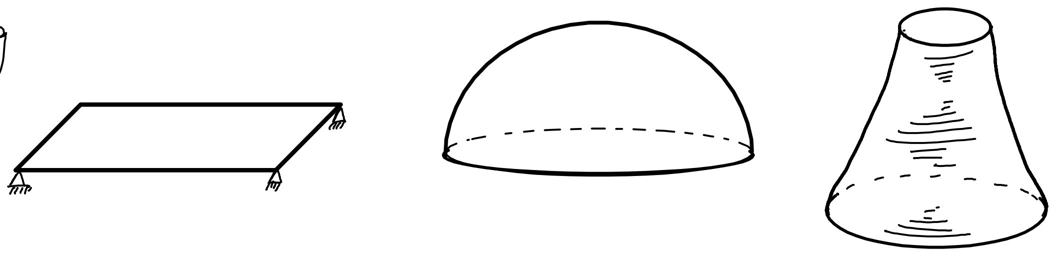
Friday, April 02, 2010
8:15 AM

Chapter 5: Plates & Shells

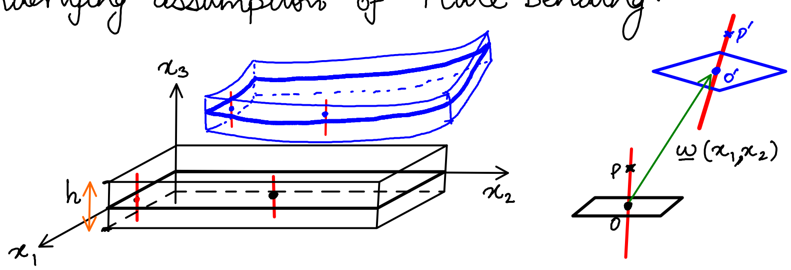
Ref: Hjelmstad Ch 8
Reddy Ch 12
Hughes Ch 5
Zienkiewicz & Taylor (Vol-2) Ch 11-12

- Plates and shells are "Structural" Theories.
- Applicable to structures with one dimension much smaller compared to the other two.

eg



Underlying assumption of Plate Bending:



$\vec{p}' \rightarrow \vec{o}' + \underline{\theta}(x_1,x_2) \times x_3\,\underline{e}_3$

$\vec{o}' \rightarrow \vec{o} + \underline{\omega}(x_1,x_2)$

$\vec{p} \rightarrow \vec{o} + x_3\,\underline{e}_3$

$\vec{o} \rightarrow \underbrace{x_1\,\underline{e}_1 + x_2\,\underline{e}_2}$

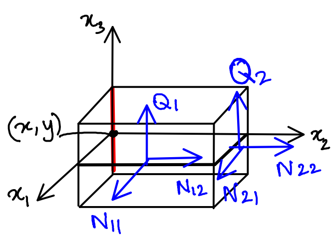
The plate deformation is completely characterized by the deformation of the reference surface (▱) and rotation of the rigid fibre (|).

$$\boxed{\underline{u}(x_1 x_2 x_3) = \underline{\omega}(x_1,x_2) + \underline{\theta}(x_1,x_2) \times \underline{p}(x_3)}$$

Define Stress Resultants:

$$\boxed{\underline{Q}_\alpha(x,y) \equiv \int_{-h/2}^{h/2} \underline{t}_\alpha\, dx_3}$$

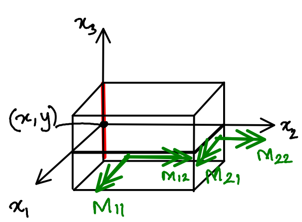
$$= \int_{-h/2}^{h/2} (\underline{\underline{\sigma}}\ \underline{e}_\alpha)\, dx_3$$



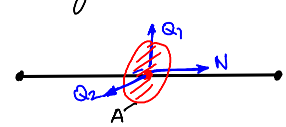
$$\underline{Q}_1 = \begin{Bmatrix} N_{11} \\ N_{12} \\ Q_1 \end{Bmatrix} \equiv \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{11} \\ \sigma_{21} \\ \sigma_{31} \end{Bmatrix} dx_3 \quad ; \quad \underline{Q}_2 = \begin{Bmatrix} N_{21} \\ N_{22} \\ Q_2 \end{Bmatrix} \equiv \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_{12} \\ \sigma_{22} \\ \sigma_{32} \end{Bmatrix} dx_3$$

$$\boxed{\underline{M}_\alpha(x,y) \equiv \int_{-h/2}^{h/2} \left[(x_3\,\underline{e}_3) \times \underline{t}_\alpha\right] dx_3}$$

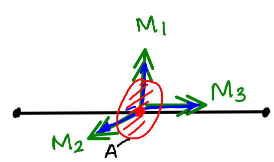
$$\underline{M}_1 = \begin{Bmatrix} M_{11} \\ M_{12} \\ 0 \end{Bmatrix} \qquad \underline{M}_2 = \begin{Bmatrix} M_{21} \\ M_{22} \\ 0 \end{Bmatrix}$$



Analogous Beam Resultants:

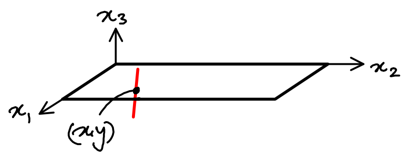


$$\underline{Q}(x_3) \equiv \int_A \underline{t}_3\, dA = \begin{Bmatrix} Q_1 \\ Q_2 \\ N \end{Bmatrix}$$



$$\underline{M}(x_3) \equiv \int_A \left[(x_1\,\underline{e}_1 + x_2\,\underline{e}_2) \times \underline{t}_3\right] dA$$

Equilibrium Equations :

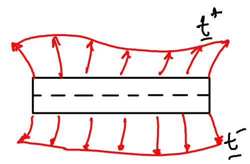


At all points (x,y) :

$$\sum \underset{\sim}{F} = 0 \Rightarrow \int_{-h/2}^{h/2} (\operatorname{div} \underset{\sim}{\sigma} + \underline{b})\, dx_3 \Rightarrow \boxed{\underline{Q}_{\alpha,\alpha} + \underline{q} = \underline{0}}$$

$$\sum \underline{M} = 0 \Rightarrow \int_{-h/2}^{h/2} x_3 \underline{e}_3 \times (\operatorname{div} \underset{\sim}{\sigma} + \underline{b})\, dx_3 \Rightarrow \boxed{\underline{M}_{\alpha,\alpha} + \underline{e}_\alpha \times \underline{Q}_\alpha + \underline{m} = \underline{0}}$$

where $\underline{q}$ & $\underline{m}$ applied loads :



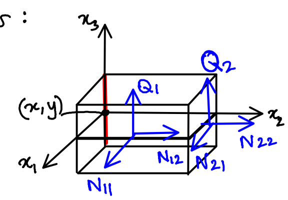
$$\underline{q} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = (\underline{t}^+ + \underline{t}^-) + \int_{-h/2}^{h/2} \underline{b}\, dx_3$$

$$\underline{m} = \begin{Bmatrix} m_1 \\ m_2 \\ 0 \end{Bmatrix} = \underline{e}_3 \times \left(\frac{h}{2} \underline{t}^+ - \frac{h}{2} \underline{t}^- \right) + \underline{e}_3 \times \int_{-h/2}^{h/2} x_3 \underline{b}\, dx_3$$

Equilibrium equations in terms of components :

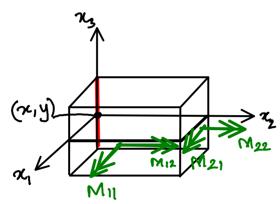
In-Plane :

$$\boxed{\begin{aligned} N_{11,1} + N_{21,2} + q_1 &= 0 \\ N_{12,1} + N_{22,2} + q_2 &= 0 \\ N_{12} &= N_{21} \end{aligned}}$$



Transverse & Bending :

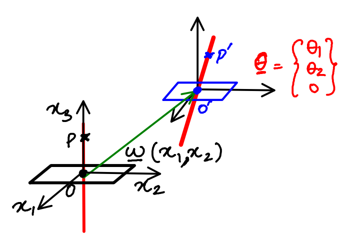
$$\boxed{\begin{aligned} Q_{1,1} + Q_{2,2} + q_3 &= 0 \\ M_{11,1} + M_{21,2} + Q_2 + m_1 &= 0 \\ M_{12,1} + M_{22,2} - Q_1 + m_2 &= 0 \end{aligned}}$$



Strain Resultants

Recall kinematic assumption :

$$\underline{u}(x_1,x_2,x_3) = \underline{w}(x_1,x_2) + \underline{\theta}(x_1,x_2) \times x_3 \underline{e}_3$$



Axial / Shear strains : $\underline{\epsilon}_\alpha = \underline{w}_{,\alpha} + \underline{e}_\alpha \times \underline{\theta}$

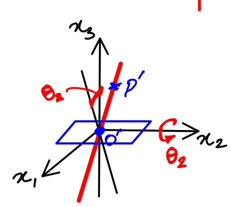
Curvatures : $\underline{\kappa}_\alpha = \underline{\theta}_{,\alpha}$

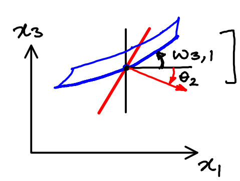
$$\underline{\epsilon}_1 = w_{1,1}\, \underline{e}_1 + w_{2,1}\, \underline{e}_2 + (w_{3,1} + \theta_2)\, \underline{e}_3$$

$$\underline{\epsilon}_2 = w_{1,2}\, \underline{e}_1 + w_{2,2}\, \underline{e}_2 + (w_{3,2} - \theta_1)\, \underline{e}_3$$

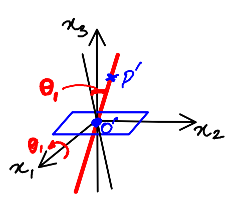
$$\underline{\kappa}_1 = \theta_{1,1}\, \underline{e}_1 + \theta_{2,1}\, \underline{e}_2$$

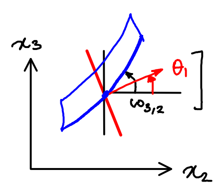
$$\underline{\kappa}_2 = \theta_{1,2}\, \underline{e}_1 + \theta_{2,2}\, \underline{e}_2$$





$\gamma = w_{3,1} + \theta_2$

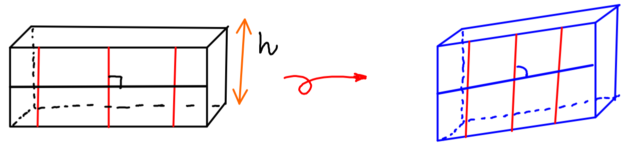




$\gamma = w_{3,2} - \theta_1$

Thin vs. Thick Plates

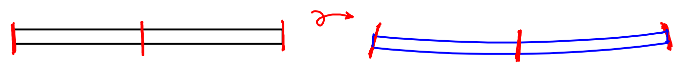
Thick plates : (Reissner–Mindlin)



- shear deformation is important.

$$\Rightarrow \quad Q_1 = Gh\,(\omega_{3,1} + \theta_2)$$
$$Q_2 = Gh\,(\omega_{3,2} - \theta_1)$$

Thin plates : (Kirchhoff–Love)



- Bending dominates & shear is negligible.
- Assume :

$$\boxed{\begin{aligned} \omega_{3,1} + \theta_2 &= 0 \\ \omega_{3,2} - \theta_1 &= 0 \end{aligned}}$$

Material Constitutive equations :

In-Plane forces:

$$N_{11} = h\left[\lambda(\omega_{1,1} + \omega_{2,2}) + 2\mu\,\omega_{1,1}\right]$$
$$N_{22} = h\left[\lambda(\omega_{1,1} + \omega_{2,2}) + 2\mu\,\omega_{2,2}\right]$$
$$N_{12} = N_{21} = h\left[\mu(\omega_{1,2} + \omega_{2,1})\right]$$

Out of plane shears :

$$Q_1 = Gh\,(\omega_{3,1} + \theta_2)$$
$$Q_2 = Gh\,(\omega_{3,2} - \theta_1)$$

(Reissner–Mindlin)

$$Q_1 = +(M_{12,1} + M_{22,2} + m_2)$$
$$Q_2 = -(M_{11,1} + M_{21,2} + m_1)$$

(Kirchhoff-Love) Rigid

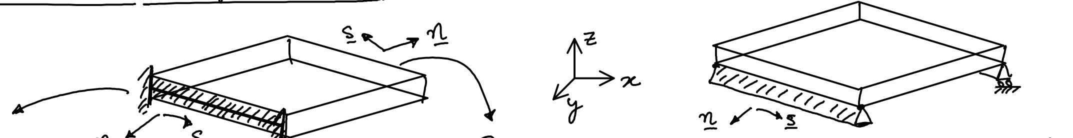
Transverse Bending :

$$M_{\alpha\beta} = \frac{h^3}{12}\left[\lambda\,\epsilon_{\alpha\beta}\,\epsilon_{\gamma\eta}\,\theta_{\eta,\gamma} + \mu\left(\epsilon_{\gamma\beta}\,\epsilon_{\alpha\eta}\,\theta_{\eta,\gamma} + \theta_{\beta,\alpha}\right)\right]$$

(Greek indices $\alpha, \beta, \eta, \gamma$: 1, 2)

(and permutation $\epsilon_{\alpha\beta}$: $\epsilon_{11} = \epsilon_{22} = 0$; $\epsilon_{12} = 1$; $\epsilon_{21} = -1$)

Boundary Conditions for Reissner–Mindlin Plates



Fixed

$\underline{\omega} = \underline{0}$ $(\omega_1 = \omega_2 = \omega_3 = 0)$

$\underline{\theta} \cdot \underline{n} = 0$

$\underline{\theta} \cdot \underline{s} = 0$

Free

$\underline{Q}_\alpha\, n_\alpha = \underline{0}$

$\underline{M}_\alpha\, n_\alpha = \underline{0}$

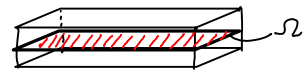
Simply supported

$\underline{\omega} = \underline{0}$

$(\underline{M}_\alpha\, n_\alpha) \cdot \underline{s} = 0$

$\underline{\theta} \cdot \underline{n} = 0$

Principle of Virtual Work for Plates:



- Reissner-Mindlin:

$$G(\{\underline{\omega},\underline{\theta}\},\{\underline{\bar\omega},\underline{\bar\theta}\}) \equiv \int_\Omega \underline{\bar\omega}\cdot(\underline{Q}_{\alpha,\alpha}+\underline{q})\,d\Omega + \int_\Omega \underline{\bar\theta}\cdot(\underline{M}_{\alpha,\alpha}+\underline{e}_\alpha\times\underline{Q}_\alpha+\underline{m})\,d\Omega$$

Integrate by parts:

$$G(\{\underline{\omega},\underline{\theta}\},\{\underline{\bar\omega},\underline{\bar\theta}\}) = -\Bigg[\int_\Omega \underbrace{(\underline{\bar\varepsilon}_\alpha\cdot\underline{Q}_\alpha + \underline{\bar\kappa}_\alpha\cdot\underline{M}_\alpha)\,d\Omega}_{W_I} \underbrace{-\int_\Omega(\underline{\bar\omega}\cdot\underline{q}+\underline{\bar\theta}\cdot\underline{m})\,d\Omega - \int_\Gamma(\underline{\bar\omega}\cdot\underline{q}_\Gamma+\underline{\bar\theta}\cdot\underline{m}_\Gamma)\,d\Gamma}_{W_E}\Bigg]$$

where virtual strains & curvatures: $\underline{\bar\varepsilon}_\alpha = \underline{\bar\omega}_{,\alpha} + \underline{e}_\alpha\times\underline{\bar\theta}_\alpha$

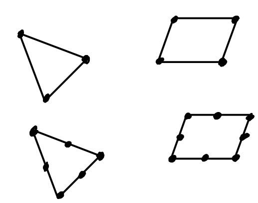
$\underline{\bar\kappa}_\alpha = \underline{\bar\theta}_{,\alpha}$

Finite Element Approximation:

Since highest order derivative in $G(\cdot,\cdot)$ is $1 \Rightarrow H_0^1(\Omega)$

Thus all our 2D finite elements : T3, Q4, T6, Q8, Q9 -------
are acceptable for

$$\underline{\omega} = \begin{Bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{Bmatrix} \quad\text{and}\quad \underline{\theta} = \begin{Bmatrix}\theta_1\\ \theta_2\end{Bmatrix}$$



However, just like Timoshenko-Beam element,
equal order interpolation of $\omega(x)$ & $\theta(x)$ leads to Shear Locking.

Once again, this issue is solved by reduced integration of the shear term:

$$\int_\Omega(\underline{\bar\omega}\cdot\underline{Q}_\alpha)\,d\Omega$$

Kirchhoff–Love Plate equations (Bending Only)

Assuming the shear strains are zero :

$$\omega_{3,1} + \theta_2 = 0$$
$$\omega_{3,2} - \theta_1 = 0$$

Let $\omega_3(x,y) = \omega(x,y)$; $q_3 = q$ and $\underline{m} = \underline{0}$

The constitutive equation for moments :

$$M_{\alpha\beta} = D\left(\nu\, \epsilon_{\beta\alpha}\, w_{,\gamma\gamma} + (1-\nu)\, \epsilon_{\beta\gamma}\, \omega_{,\alpha\gamma}\right)$$

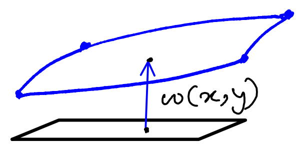
where plate modulus $D = \dfrac{E h^3}{12(1-\nu^2)}$

Equilibrium for shears :

$$Q_1 = -D\left(\omega_{,111} + \omega_{,221}\right)$$
$$Q_2 = -D\left(\omega_{,222} + \omega_{,112}\right)$$

Substituting into equilibrium for moments :

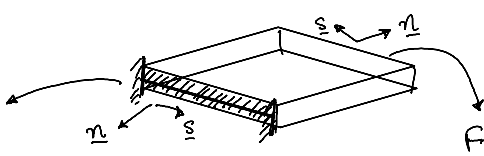
$$\boxed{D\, \nabla^4 \omega = q}$$

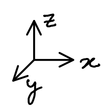


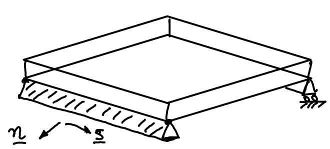
i.e.

$$D\left(\frac{\partial^4 \omega}{\partial x^4} + 2\frac{\partial^4 \omega}{\partial x^2 \partial y^2} + \frac{\partial^4 \omega}{\partial y^4}\right) = q$$

Boundary Conditions for Kirchhoff-Love Plates







Fixed

$\omega = 0$

$(\nabla \underline{\omega}) \cdot \underline{n} = 0$

Free

$(\underline{M}_\alpha n_\alpha) \cdot \underline{s} = 0$

$$\boxed{Q_\alpha n_\alpha - \nabla(\underline{m}_p \cdot \underline{n}) \cdot \underline{s} = \underline{0}}$$

Effective shear

(Kirchhoff-free edge condition)

Simply supported

$\omega = 0$

$(\underline{M}_\alpha n_\alpha) \cdot \underline{s} = 0$

Principle of Virtual Work for Kirchhoff-Love Plates:

(Bending Only)

$$G(w, \bar{w}) = \int_\Omega \bar{w}\left(D\nabla^4 w - q\right) d\Omega$$

Integration by parts twice:

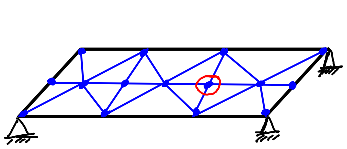
$$\boxed{G(w, \bar{w}) = \int_\Omega \left[D \left[\nu\, \bar{w}_{,\alpha\alpha}\, w_{,\beta\beta} + (1-\nu)\, \bar{w}_{,\alpha\beta}\, w_{,\alpha\beta} \right] - \bar{w}\, q \right] d\Omega + \text{Boundary Terms}}$$

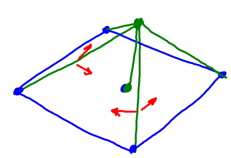
Finite Element Approximation

Note, that the highest order derivative in $G(\cdot,\cdot)$ is 2

Thus, our approximation functions must be in $H_0^2(\Omega)$

This means:



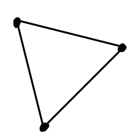


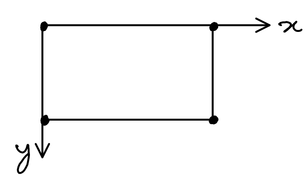
(Derivative discontinuous)

2D Lagrange shape functions are not ok.

If each node has dofs:

$w, \dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial y}$ (Bending)



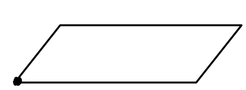


Then it is not possible to enforce compatibility in general.

Still, incompatible elements are used often and give good results.

One can construct compatible elements by including higher order derivatives as dofs at nodes.

$w, \dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial y}, \dfrac{\partial^2 w}{\partial x \partial y}$ eg



(Bending only)

This remains an active area of research.