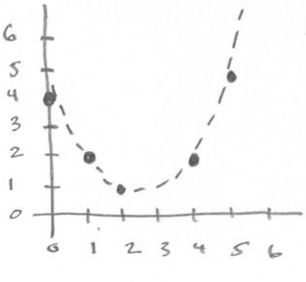
CE506 Homework 3 Solution 21 Sept 2003, problem 1.



Fit parabola to 5 points, parabola: $\boxed{y = a_0 + a_1 x + a_2 x^2}$

x	y
0	4
1	2
2	1
4	2
5	5

$n = 5$
$n_0 = 3$
$r = 2$

use the least squares technique of observations only, question: how to generate the condition equations? let's try elimination approach,

write 4 equations with 3 parameters then eliminate those 3 parameters,

$$\hat{y}_1 = a_0$$
$$\hat{y}_2 = a_0 + a_1 + a_2$$
$$\hat{y}_3 = a_0 + 2a_1 + 4a_2$$
$$\hat{y}_4 = a_0 + 4a_1 + 16a_2$$

solve first equation for a_0 and substitute

$$\hat{y}_2 = \hat{y}_1 + a_1 + a_2$$
$$\hat{y}_3 = \hat{y}_1 + 2a_1 + 4a_2$$
$$\hat{y}_4 = \hat{y}_1 + 4a_1 + 16a_2$$

solve first equation for a_1 and substitute

$(a_1 = \hat{y}_2 - \hat{y}_1 - a_2)$

$$\hat{y}_3 = -\hat{y}_1 + 2\hat{y}_2 + 12a_2$$
$$\hat{y}_4 = -3\hat{y}_1 + 4\hat{y}_2 + 12a_2$$

solve first equation for $2a_2$ and substitute

$(2a_2 = \hat{y}_3 + \hat{y}_1 - 2\hat{y}_2)$

$$\boxed{3\hat{y}_1 - 8\hat{y}_2 + 6\hat{y}_3 - \hat{y}_4 = 0}$$

plug in the observations, y, and we get

$$\boxed{3v_1 - 8v_2 + 6v_3 - v_4 = 0}$$

that's one condition equation

we did it with equations 1,2,3,4 now do with equations 1,2,3,5

$$\hat{y}_1 = a_0$$
$$\hat{y}_2 = a_0 + a_1 + a_2$$
$$\hat{y}_3 = a_0 + 2a_1 + 4a_2$$
$$\hat{y}_5 = a_0 + 5a_1 + 25a_2$$

solve first equation for a_0 and substitute

$$\hat{y}_2 = \hat{y}_1 + a_1 + a_2$$
$$\hat{y}_3 = \hat{y}_1 + 2a_1 + 4a_2$$
$$\hat{y}_5 = \hat{y}_1 + 5a_1 + 25a_2$$

solve first equation for a_1 and substitute

$(a_1 = \hat{y}_2 - \hat{y}_1 - a_2)$

$$\hat{y}_3 = -\hat{y}_1 + 2\hat{y}_2 + 2a_2$$
$$\hat{y}_5 = -4\hat{y}_1 + 5\hat{y}_2 + 20a_2$$

solve first equation for $2a_2$ and substitute

$$2a_2 = \hat{y}_3 + \hat{y}_1 - 2\hat{y}_2$$

$$\boxed{6\hat{y}_1 - 15\hat{y}_2 + 10\hat{y}_3 - \hat{y}_5 = 0}$$

plug in the observations, y, and we get

$$\boxed{6v_1 - 15v_2 + 10v_3 - v_5 = 1}$$

there is the 2nd condition equation

Note 1. If I evaluate the determinants as in the homework hint, I get equivalent result,

$$\begin{vmatrix} \hat{y}_1 & 0 & 0 & 1 \\ \hat{y}_2 & 1 & 1 & 1 \\ \hat{y}_3 & 4 & 2 & 1 \\ \hat{y}_4 & 16 & 4 & 1 \end{vmatrix} = 0,$$

$$-6\hat{y}_1 + 16\hat{y}_2 - 12\hat{y}_3 + 2\hat{y}_4 = 0$$

that is just (-2) times the first equation above, so it is same

$$\begin{vmatrix} \hat{y}_1 & 0 & 0 & 1 \\ \hat{y}_2 & 1 & 1 & 1 \\ \hat{y}_3 & 4 & 2 & 1 \\ \hat{y}_5 & 25 & 5 & 1 \end{vmatrix} = 0$$

$$-12\hat{y}_1 + 30\hat{y}_2 - 20\hat{y}_3 + 2\hat{y}_5 = 0$$

that is also (-2) times the second equation so it is same.

these determinants are evaluations of

$$\begin{vmatrix} y_1 & x_1^2 & x_1 & 1 \\ y_2 & x_2^2 & x_2 & 1 \\ y_3 & x_3^2 & x_3 & 1 \\ y_4 & x_4^2 & x_4 & 1 \end{vmatrix} = 0$$, etc. for any 4 points on the curve.

Note 2. For the elimination procedure I could have started with 5 equations, eliminated 3 unknowns (a_0, a_1, a_2) and I would have been left with same 2 equations (this would have been shorter!)

---

now we have 2 condition equations, let's solve the LS problem, using Lagrange

$$\phi' = v_1^2 + v_2^2 + v_3^2 + v_4^2 + v_5^2 + 2k_1(3v_1 - 8v_2 + 6v_3 - v_4) + 2k_2(6v_1 - 15v_2 + 10v_3 - v_5 - 1)$$

$$\partial\phi'/\partial v_1 = \cancel{2}v_1 + \cancel{2}\cdot 3k_1 + \cancel{2}\cdot 6k_2 = 0$$
$$\partial\phi'/\partial v_2 = \cancel{2}v_2 - \cancel{2}\cdot 8k_1 - \cancel{2}\cdot 15k_2 = 0$$
$$\partial\phi'/\partial v_3 = \cancel{2}v_3 + \cancel{2}\cdot 6k_1 + \cancel{2}\cdot 10k_2 = 0$$
$$\partial\phi'/\partial v_4 = \cancel{2}v_4 - \cancel{2}\cdot k_1 = 0$$
$$\partial\phi'/\partial v_5 = \cancel{2}v_5 - \cancel{2}\cdot k_2 = 0$$
$$\partial\phi'/\partial k_1 = \cancel{2}(3v_1 - 8v_2 + 6v_3 - v_4) = 0$$
$$\partial\phi'/\partial k_2 = \cancel{2}(6v_1 - 15v_2 + 10v_3 - v_5 - 1) = 0$$

$$\Rightarrow \left[\begin{array}{ccccc|cc} 1 & 0 & 0 & 0 & 0 & 3 & 6 \\ 0 & 1 & 0 & 0 & 0 & -8 & -15 \\ 0 & 0 & 1 & 0 & 0 & 6 & 10 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ \hline 3 & -8 & 6 & -1 & 0 & 0 & 0 \\ 6 & -15 & 10 & 0 & -1 & 0 & 0 \end{array}\right] \left[\begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ \hline k_1 \\ k_2 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ 1 \end{array}\right]$$

Extract as matrix equation, partitions reveal the structure of the matrix equation

solve the system,
$$\left[\begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ \hline k_1 \\ k_2 \end{array}\right] = \left[\begin{array}{c} .1071 \\ -.1071 \\ -.1429 \\ .3214 \\ -.1786 \\ \hline .3214 \\ -.1786 \end{array}\right] \qquad \hat{y} = y + v, \quad \hat{y} = \begin{bmatrix} 4.1071 \\ 1.8929 \\ .8571 \\ 2.3214 \\ 4.8214 \end{bmatrix}$$

Note 3. look at these residuals (corrections) under equal weight assumption and compare to the residuals in the next problem. What is the influence of the weights?

2. Solve the same problem as #1, but with
(a) indirect observation method, and
(b) observation weights of 3, 2, 1, 2, 3 for y_1, y_2, y_3, y_4, y_5

$$\left.\begin{array}{l} n=5 \\ n_0=3 \\ \hline r=2 \end{array}\right\}$$ choose $n_0 = u$ parameters: a_0, a_1, a_2 $\quad \hat{y}_i = a_0 + a_1 x_i + a_2 x_i^2$

write $n=5$ condition equations expressing each observation as a function of the chosen parameters,

$$\begin{array}{l|l}
\hat{y}_1 = y_1 + v_1 = 4 + v_1 = a_0 + a_1(0) + a_2(0)^2 & v_1 = a_0 - 4 \\
\hat{y}_2 = y_2 + v_2 = 2 + v_2 = a_0 + a_1(1) + a_2(1)^2 & v_2 = a_0 + a_1 + a_2 - 2 \\
\hat{y}_3 = y_3 + v_3 = 1 + v_3 = a_0 + a_1(2) + a_2(2)^2 & v_3 = a_0 + 2a_1 + 4a_2 - 1 \\
\hat{y}_4 = y_4 + v_4 = 2 + v_4 = a_0 + a_1(4) + a_2(4)^2 & v_4 = a_0 + 4a_1 + 16a_2 - 2 \\
\hat{y}_5 = y_5 + v_5 = 5 + v_5 = a_0 + a_1(5) + a_2(5)^2 & v_5 = a_0 + 5a_1 + 25a_2 - 5
\end{array}$$

put weights into the objective function: $\Phi = 3v_1^2 + 2v_2^2 + v_3^2 + 2v_4^2 + 3v_5^2$
now substitute expressions for v_i into the objective function

$$\Phi = \underset{(w_1)}{3}(a_0-4)^2 + \underset{(w_2)}{2}(a_0+a_1+a_2-2)^2 + \underset{(w_3=1)}{}(a_0+2a_1+4a_2-1)^2 + \underset{(w_4)}{2}(a_0+4a_1+16a_2-2)^2 +$$

$$\underset{(w_5)}{3}(a_0+5a_1+25a_2-5)^2$$, now differentiate with respect to a_0, a_1, a_2 (remember the chain rule)

$$\partial\Phi/\partial a_0 = \cancel{2}\cdot 3(a_0-4) + \cancel{2}\cdot 2(a_0+a_1+a_2-2) + \cancel{2}(a_0+2a_1+4a_2-1) +$$
$$\cancel{2}\cdot 2(a_0+4a_1+16a_2-2) + \cancel{2}\cdot 3(a_0+5a_1+25a_2-5) = 0$$

$$\partial\Phi/\partial a_1 = 0 + \cancel{2}\cdot 2(a_0+a_1+a_2-2) + \cancel{2}(a_0+2a_1+4a_2-1)2 +$$
$$\cancel{2}\cdot 2(a_0+4a_1+16a_2-2)4 + \cancel{2}\cdot 3(a_0+5a_1+25a_2-5)5 = 0$$

$$\partial\Phi/\partial a_2 = 0 + \cancel{2}\cdot 2(a_0+a_1+a_2-2) + \cancel{2}(a_0+2a_1+4a_2-1)4 +$$
$$\cancel{2}\cdot 2(a_0+4a_1+16a_2-2)16 + \cancel{2}\cdot 3(a_0+5a_1+25a_2-5)25 = 0$$

collect like terms, and extract the coefficient matrix

$$11a_0 + 27a_1 + 113a_2 = 36$$
$$27a_0 + 113a_1 + 513a_2 = 97$$
$$113a_0 + 513a_1 + 2405a_2 = 447$$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 4.0852 \\ -2.8482 \\ 0.6015 \end{bmatrix}$$ for reference, the equally weighted solution: $$\begin{bmatrix} 4.1071 \\ -2.8036 \\ 0.5893 \end{bmatrix}$$

$$\begin{bmatrix} 11 & 27 & 113 \\ 27 & 113 & 513 \\ 113 & 513 & 2405 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 36 \\ 97 \\ 447 \end{bmatrix}$$

plug the a_0, a_1, a_2 values back into equations for v_i, then add $\hat{y}_i = y_i + v_i$

$$v = \begin{bmatrix} .0852 \\ -.1616 \\ -.2054 \\ .3157 \\ -.1194 \end{bmatrix}, \quad \hat{y} = \begin{bmatrix} 4.0852 \\ 1.8384 \\ 0.7946 \\ 2.3157 \\ 4.8806 \end{bmatrix}$$

3. Fit plane with model $\boxed{z = a_0 + a_1 x + a_2 y}$ to the following data

x	y	z
1	1	3.0
1	-1	2.0
-1	1	1.5
-1	-1	1.2
2	1	2.8

x, y constant, z observation, solve the LS problem by the method of <u>indirect observations</u> $(W=I)$

$n=5$
$n_0=3$
$r=2$

choose $n_0 = u$ parameters a_0, a_1, a_2

write $n=5$ condition equations, expressing each adjusted observation as a function of the parameters,

$$\hat{z}_1 = z_1 + v_1 = 3 + v_1 = a_0 + a_1(1) + a_2(1) \qquad v_1 = a_0 + a_1 + a_2 - 3$$
$$\hat{z}_2 = z_2 + v_2 = 2 + v_2 = a_0 + a_1(1) - a_2(1) \qquad v_2 = a_0 + a_1 - a_2 - 2$$
$$\hat{z}_3 = z_3 + v_3 = 1.5 + v_3 = a_0 - a_1(1) + a_2(1) \qquad v_3 = a_0 - a_1 + a_2 - 1.5$$
$$\hat{z}_4 = z_4 + v_4 = 1.2 + v_4 = a_0 - a_1(1) - a_2(1) \qquad v_4 = a_0 - a_1 - a_2 - 1.2$$
$$\hat{z}_5 = z_5 + v_5 = 2.8 + v_5 = a_0 + a_2(2) + a_2(1) \qquad v_5 = a_0 + 2a_1 + a_2 - 2.8$$

substitute expressions for v_i into the objective function $\Phi = v_1^2 + v_2^2 + v_3^2 + v_4^2 + v_5^2$

$$\Phi = (a_0 + a_1 + a_2 - 3)^2 + (a_0 + a_1 - a_2 - 2)^2 + (a_0 - a_1 + a_2 - 1.5)^2 + (a_0 - a_1 - a_2 - 1.2)^2 + (a_0 + 2a_1 + a_2 - 2.8)^2$$

, now differentiate with respect to 3 parameters,

$$\partial\Phi/\partial a_0 = 2(a_0 + a_1 + a_2 - 3) + 2(a_0 + a_1 - a_2 - 2) + 2(a_0 - a_1 + a_2 - 1.5) + 2(a_0 - a_1 - a_2 - 1.2) + 2(a_0 + 2a_1 + a_2 - 2.8) = 0$$

$$\partial\Phi/\partial a_1 = 2(a_0 + a_1 + a_2 - 3) + 2(a_0 + a_1 - a_2 - 2) + 2(a_0 - a_1 + a_2 - 1.5)(-1) + 2(a_0 - a_1 - a_2 - 1.2)(-1) + 2(a_0 + 2a_1 + a_2 - 2.8)(2) = 0$$

$$\partial\Phi/\partial a_2 = 2(a_0 + a_1 + a_2 - 3) + 2(a_0 + a_1 - a_2 - 2)(-1) + 2(a_0 - a_1 + a_2 - 1.5) + 2(a_0 - a_1 - a_2 - 1.2)(-1) + 2(a_0 + 2a_1 + a_2 - 2.8) = 0$$

collect terms and extract the matrix equation,

$$5a_0 + 2a_1 + a_2 = 10.5$$
$$2a_0 + 8a_1 + 2a_2 = 7.9$$
$$a_0 + 2a_1 + 5a_2 = 4.1$$

$$\begin{bmatrix} 5 & 2 & 1 \\ 2 & 8 & 2 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 10.5 \\ 7.9 \\ 4.1 \end{bmatrix} \qquad \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1.8650 \\ 0.4550 \\ 0.2650 \end{bmatrix}$$

subsitute to get v then get $\hat{z} = z + v$

$$V = \begin{bmatrix} -.4150 \\ .0550 \\ .1750 \\ -.0550 \\ .2400 \end{bmatrix}, \quad \hat{z} = \begin{bmatrix} 2.5850 \\ 2.0550 \\ 1.6750 \\ 1.1450 \\ 3.0400 \end{bmatrix}$$