

Derivation of Matrix Solution to Nonlinear Least Squares by Indirect Observations

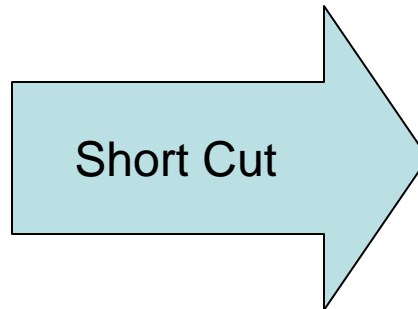
n : observations, n_0 : minimum to define model, r : redundancy, $u=n_0$: parameters

$$l_1 = F_1(x_1, x_2, \dots, x_u)$$

$$l_2 = F_2(x_1, x_2, \dots, x_u)$$

⋮

$$l_n = F_n(x_1, x_2, \dots, x_u)$$



$$\mathbf{J} = \mathbf{J}_F = \frac{\partial \mathbf{F}}{\partial \mathbf{x}}$$

$$(\mathbf{l} - \mathbf{F}^0) + \mathbf{v} = \mathbf{J}_F \Delta$$

$$\mathbf{f} = \mathbf{l} - \mathbf{F}^0$$

$$\mathbf{f} + \mathbf{v} = \mathbf{J} \Delta$$

n equations, u unknown parameters,
one equation per observation

$$G_1 = l_1 - F_1(x_1, x_2, \dots, x_u)$$

$$G_2 = l_2 - F_2(x_1, x_2, \dots, x_u)$$

⋮

$$G_n = l_n - F_n(x_1, x_2, \dots, x_u)$$

$$\Delta = (\mathbf{J}^T \mathbf{W} \mathbf{J})^{-1} \mathbf{J}^T \mathbf{W} \mathbf{f}$$

$$\mathbf{v} = \mathbf{J} \Delta - \mathbf{f}$$

or,

$$\Delta = (\mathbf{A}^T \mathbf{W} \mathbf{A})^{-1} \mathbf{A}^T \mathbf{W} \mathbf{f}, \text{ etc.}$$

Linearize by Taylor Series

$$\begin{aligned}0 &= G_1(l_1, x_1, x_2, \dots, x_u) \approx G_1(l_1^o, x_1^o, x_2^o, \dots, x_u^o) + \frac{\partial G_1}{\partial l_1} \Delta l_1 + \frac{\partial G_1}{\partial x_1} \Delta x_1 + \dots + \frac{\partial G_1}{\partial x_u} \Delta x_u \\0 &= G_2(l_2, x_1, x_2, \dots, x_u) \approx G_2(l_2^o, x_1^o, x_2^o, \dots, x_u^o) + \frac{\partial G_2}{\partial l_2} \Delta l_2 + \frac{\partial G_2}{\partial x_1} \Delta x_1 + \dots + \frac{\partial G_2}{\partial x_u} \Delta x_u \\&\vdots \\0 &= G_n(l_n, x_1, x_2, \dots, x_u) \approx G_n(l_n^o, x_1^o, x_2^o, \dots, x_u^o) + \frac{\partial G_n}{\partial l_n} \Delta l_n + \frac{\partial G_n}{\partial x_1} \Delta x_1 + \dots + \frac{\partial G_n}{\partial x_u} \Delta x_u\end{aligned}$$

Rearranging into compact vector / matrix notation,

$$\begin{bmatrix} -G_1^o \\ -G_2^o \\ \vdots \\ -G_n^o \end{bmatrix} = \mathbf{I}_n \begin{bmatrix} \Delta l_1 \\ \Delta l_2 \\ \vdots \\ \Delta l_n \end{bmatrix} + \mathbf{J}_G \Delta \quad \text{note : } \mathbf{J}_G = -\mathbf{J}_F$$

$$-\mathbf{G}^o = \Delta \mathbf{l} + \mathbf{J}_G \Delta$$

$$-\mathbf{G}^0 = \Delta \mathbf{l} + \mathbf{J}_G \Delta$$

$$\mathbf{G}^0 = \mathbf{l}^0 - \mathbf{F}^0$$

$$\mathbf{l} + \mathbf{v} = \mathbf{l}^0 + \Delta \mathbf{l}$$

$$\Delta \mathbf{l} = \mathbf{l} + \mathbf{v} - \mathbf{l}^0$$

substituting these into prior expression

$$\mathbf{F}^0 - \mathbf{l}^0 = \mathbf{l} + \mathbf{v} - \mathbf{l}^0 + \mathbf{J}_G \Delta$$

rearrange and simplify

$$-\mathbf{l} + \mathbf{F}^0 - \mathbf{v} = \mathbf{J}_G \Delta = -\mathbf{J}_F \Delta$$

$$\left(\mathbf{l} - \mathbf{F}^0\right) + \mathbf{v} = \mathbf{J}_F \Delta$$

Then follow the “short cut” on the first page of this presentation