Derivation of Matrix Solution to Nonlinear Least Squares by Indirect Observations

n: observations, n_o: minimum to define model, r: redundancy, $u=n_o$: parameters

$$l_1 = F_1(x_1, x_2, \cdots, x_u)$$
$$l_2 = F_2(x_1, x_2, \cdots, x_u)$$
$$\vdots$$
$$l_n = F_n(x_1, x_2, \cdots, x_u)$$

n equations, u unknown parameters,
one equation per observation

$$G_1 = l_1 - F_1(x_1, x_2, \cdots, x_u)$$
$$G_2 = l_2 - F_2(x_1, x_2, \cdots, x_u)$$
$$\vdots$$
$$G_n = l_n - F_n(x_1, x_2, \cdots, x_u)$$

Short Cut

$$\mathbf{J} = \mathbf{J_F} = \frac{\partial \mathbf{F}}{\partial \mathbf{x}}$$
$$(\mathbf{l} - \mathbf{F^o}) + \mathbf{v} = \mathbf{J_F}\Delta$$
$$\mathbf{f} = \mathbf{l} - \mathbf{F^o}$$
$$\mathbf{f} + \mathbf{v} = \mathbf{J}\Delta$$

$$\Delta = (\mathbf{J^T W J})^{-1} \mathbf{J^T W f}$$
$$\mathbf{v} = \mathbf{J}\Delta - \mathbf{f}$$
$$or,$$
$$\Delta = (\mathbf{A^T W A})^{-1} \mathbf{A^T W f}, \text{ etc.}$$

Linearize by Taylor Series

$$0 = G_1\left(l_1, x_1, x_2, \cdots, x_u\right) \approx G_1\left(l_1^o, x_1^o, x_2^o, \cdots, x_u^o\right) + \frac{\partial G_1}{\partial l_1}\Delta l_1 + \frac{\partial G_1}{\partial x_1}\Delta x_1 + \cdots + \frac{\partial G_1}{\partial x_u}\Delta x_u$$

$$0 = G_2\left(l_2, x_1, x_2, \cdots, x_u\right) \approx G_2\left(l_2^o, x_1^o, x_2^o, \cdots, x_u^o\right) + \frac{\partial G_2}{\partial l_2}\Delta l_2 + \frac{\partial G_2}{\partial x_1}\Delta x_1 + \cdots + \frac{\partial G_2}{\partial x_u}\Delta x_u$$

$$\vdots$$

$$0 = G_n\left(l_n, x_1, x_2, \cdots, x_u\right) \approx G_n\left(l_n^o, x_1^o, x_2^o, \cdots, x_u^o\right) + \frac{\partial G_n}{\partial l_n}\Delta l_n + \frac{\partial G_n}{\partial x_1}\Delta x_1 + \cdots + \frac{\partial G_n}{\partial x_u}\Delta x_u$$

Rearranging into compact vector / matrix notation,

$$\begin{bmatrix} -G_1^o \\ -G_2^o \\ \vdots \\ -G_n^o \end{bmatrix} = \mathbf{I_n} \begin{bmatrix} \Delta l_1 \\ \Delta l_2 \\ \vdots \\ \Delta l_n \end{bmatrix} + \mathbf{J_G}\Delta \qquad \text{note} : \mathbf{J_G} = -\mathbf{J_F}$$

$$-\mathbf{G^o} = \Delta\mathbf{l} + \mathbf{J_G}\Delta$$

$$-\mathbf{G^o} = \Delta\mathbf{l} + \mathbf{J_G}\Delta$$

$$\mathbf{G^o} = \mathbf{l^o} - \mathbf{F^o}$$

$$\mathbf{l} + \mathbf{v} = \mathbf{l^o} + \Delta\mathbf{l}$$

$$\Delta\mathbf{l} = \mathbf{l} + \mathbf{v} - \mathbf{l^o}$$

substituting these into prior expression

$$\mathbf{F^o} - \mathbf{l^o} = \mathbf{l} + \mathbf{v} - \mathbf{l^o} + \mathbf{J_G}\Delta$$

rearrange and simplify

$$-\mathbf{l} + \mathbf{F^o} - \mathbf{v} = \mathbf{J_G}\Delta = -\mathbf{J_F}\Delta$$

$$\left(\mathbf{l} - \mathbf{F^o}\right) + \mathbf{v} = \mathbf{J_F}\Delta$$

Then follow the “short cut” on the first page of this presentation