1. $y = a_0 + a_1 x + a_2 x^2$

x	-4.8	-2.5	1.2	4.9	6.0	6.6
y	7.2	4.9	4.2	6.9	9.6	12.8

$n = 6$

$n_0 = 3 \rightarrow a_0, a_1, a_2 = \text{parameters}$ <u>Indirect Observations</u>

$r = 3$

$c = 6$ $\quad y_i + v_i = a_0 + a_1 x_i + a_2 x_i^2, \quad v_i = a_0 + a_1 x_i + a_2 x_i^2 - y_i$

$\Phi = v_1^2 + v_2^2 + v_3^2 + v_4^2 + v_5^2 + v_6^2$

$$\phi = [a_0 + a_1(-4.8) + a_2(-4.8)^2 - 7.2]^2 + [a_0 + a_1(-2.5) + a_2(-2.5)^2 - 4.9]^2 + [a_0 + a_1(1.2) + a_2(1.2)^2 - 4.2]^2 + [a_0 + a_1(4.9) + a_2(4.9)^2 - 6.9]^2 + [a_0 + a_1(6.0) + a_2(6.0)^2 - 9.6]^2 + [a_0 + a_1(6.6) + a_2(6.6)^2 - 12.8]^2$$

$$\frac{\partial\phi}{\partial a_0} = \cancel{2}[a_0 + a_1(-4.8) + a_2(-4.8)^2 - 7.2] + \cancel{2}[a_0 + a_1(-2.5) + a_2(-2.5)^2 - 4.9] + \cancel{2}[a_0 + a_1(1.2) + a_2(1.2)^2 - 4.2] + \cancel{2}[a_0 + a_1(4.9) + a_2(4.9)^2 - 6.9] + \cancel{2}[a_0 + a_1(6.0) + a_2(6.0)^2 - 9.6] + \cancel{2}[a_0 + a_1(6.6) + a_2(6.6)^2 - 12.8] = 0$$

$$\frac{\partial\phi}{\partial a_1} = \cancel{2}[a_0 + a_1(-4.8) + a_2(-4.8)^2 - 7.2](-4.8) + \cancel{2}[a_0 + a_1(-2.5) + a_2(-2.5)^2 - 4.9](-2.5) + \cancel{2}[a_0 + a_1(1.2) + a_2(1.2)^2 - 4.2](1.2) + \cancel{2}[a_0 + a_1(4.9) + a_2(4.9)^2 - 6.9](4.9) + \cancel{2}[a_0 + a_1(6.0) + a_2(6.0)^2 - 9.6](6.0) + \cancel{2}[a_0 + a_1(6.6) + a_2(6.6)^2 - 12.8](6.6) = 0$$

$$\frac{\partial\phi}{\partial a_2} = \cancel{2}[a_0 + a_1(-4.8) + a_2(-4.8)^2 - 7.2](-4.8)^2 + \cancel{2}[a_0 + a_1(-2.5) + a_2(-2.5)^2 - 4.9](-2.5)^2 + \cancel{2}[a_0 + a_1(1.2) + a_2(1.2)^2 - 4.2](1.2)^2 + \cancel{2}[a_0 + a_1(4.9) + a_2(4.9)^2 - 6.9](4.9)^2 + \cancel{2}[a_0 + a_1(6.0) + a_2(6.0)^2 - 9.6](6.0)^2 + \cancel{2}[a_0 + a_1(6.6) + a_2(6.6)^2 - 12.8](6.6)^2 = 0$$

$$[1+1+1+1+1+1]a_0 + [-4.8 - 2.5 + 1.2 + 4.9 + 6.0 + 6.6]a_1 + [(-4.8)^2 + (-2.5)^2 + (1.2)^2 + (4.9)^2 + (6.0)^2 + (6.6)^2]a_2 = 7.2 + 4.9 + 4.2 + 6.9 + 9.6 + 12.8$$

$$[-4.8 - 2.5 + 1.2 + 4.9 + 6.0 + 6.6]a_0 + [(-4.8)^2 + (-2.5)^2 + (1.2)^2 + (4.9)^2 + (6.0)^2 + (6.6)^2]a_1 + [(-4.8)^3 + (-2.5)^3 + (1.2)^3 + (4.9)^3 + (6.0)^3 + (6.6)^3]a_2 =$$

$$(-7.2)(4.8) - (4.9)(2.5) + (4.2)(1.2) + (6.9)(4.9) + (9.6)(6.0) + (12.8)(6.6)$$

$$\left[(-4.8)^2 + (-2.5)^2 + (1.2)^2 + (4.9)^2 + (6.0)^2 + (6.6)^2\right] a_0 +$$
$$\left[(-4.8)^3 + (-2.5)^3 + (1.2)^3 + (4.9)^3 + (6.0)^3 + (6.6)^3\right] a_1 +$$
$$\left[(-4.8)^4 + (-2.5)^4 + (1.2)^4 + (4.9)^4 + (6.0)^4 + (6.6)^4\right] a_2 =$$
$$(7.2)(-4.8)^2 + (4.9)(-2.5)^2 + (4.2)(1.2)^2 + (6.9)(4.9)^2 + (9.6)(6.0)^2 + (12.8)(6.6)^2$$

$$\left.\begin{aligned} 6a_0 + 11.4a_1 + 134.3a_2 &= 45.6 \\ 11.4a_0 + 134.3a_1 + 496.656a_2 &= 134.12 \\ 134.3a_0 + 496.656 + 4341.9314a_2 &= 1271.398 \end{aligned}\right\}$$

put in matrix form then solve in matlab.

(solve via Matlab)

$$\begin{bmatrix} 6 & 11.4 & 134.3 \\ 11.4 & 134.3 & 496.656 \\ 134.3 & 496.656 & 4341.9314 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 45.6 \\ 134.12 \\ 1271.398 \end{bmatrix}, \quad \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 3.466106 \\ 0.031265 \\ 0.182032 \end{bmatrix}$$

Subst. parameters into $V_i = a_0 + a_1 x_i + a_2 x_i^2 - y_i$

$V_1 = 0.310$ $\quad V_4 = 1.090$

$V_2 = -0.374$ $\quad V_5 = 0.607$

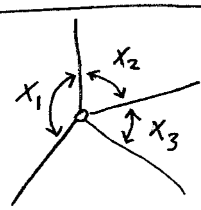
$V_3 = -0.434$ $\quad V_6 = -1.198$

---

2 (a) $n = 7$

$n_0 = 3 \quad x_1, x_2, x_3$

$r = 4$



$$\phi = V_1^2 + V_2^2 + V_3^2 + V_4^2 + V_5^2 + V_6^2 + V_7^2$$

<u>Indirect</u> <u>Observations</u>

$l_1 + V_1 = x_1 \qquad V_1 = x_1 - 120$

$l_2 + V_2 = x_2 \qquad V_2 = x_2 - 82$

$l_3 + V_3 = x_3 \qquad V_3 = x_3 - 40$

$l_4 + V_4 = 360 - x_1 - x_2 \qquad V_4 = 360 - x_1 - x_2 - 166 = -x_1 - x_2 + 194$

$l_5 + V_5 = x_1 + x_2 \qquad V_5 = x_1 + x_2 - 203$

$l_6 + V_6 = 360 - x_1 \qquad V_6 = 360 - x_1 - 238 = -x_1 + 122$

$l_7 + V_7 = x_3 \qquad V_7 = x_3 - 41$

$$\phi = (x_1 - 120)^2 + (x_2 - 82)^2 + (x_3 - 40)^2 + (-x_1 - x_2 + 194)^2 + (x_1 + x_2 - 203)^2 + (-x_1 + 122)^2 + (x_3 - 41)^2$$

$$\frac{\partial \phi}{\partial x_1} = \cancel{2}(x_1-20) + \cancel{2}(-x_1-x_2+194)(-1) + \cancel{2}(x_1+x_2-203) + \cancel{2}(-x_1+122)(-1) = 0$$

$$\frac{\partial \phi}{\partial x_2} = \cancel{2}(x_2-82) + \cancel{2}(-x_1-x_2+194)(-1) + \cancel{2}(x_1+x_2-203) = 0$$

$$\frac{\partial \phi}{\partial x_3} = \cancel{2}(x_3-40) + \cancel{2}(x_3-41) = 0$$

$$4x_1 + 2x_2 + 0x_3 = 120 + 194 + 203 + 122 = 639$$

$$2x_1 + 3x_2 + 0x_3 = 82 + 194 + 203 = 479$$

$$0x_1 + 0x_2 + 2x_3 = 40 + 41 = 81$$

$$\begin{bmatrix} 4 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 639 \\ 479 \\ 81 \end{bmatrix}, \quad \text{solve by Matlab} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 119.875 \\ 79.75 \\ 40.5 \end{bmatrix}$$

plug back into equations $v_i = \dots$

$v_1 = -.125$ $\quad$ $v_4 = -5.625$ $\quad$ $v_7 = -0.5$

$v_2 = -2.25$ $\quad$ $v_5 = -3.375$

$v_3 = 0.5$ $\quad$ $v_6 = 2.125$

2(b) $n = 7$ $\quad$ Observations only & Lagrange Multipliers

$n_0 = 3$ $\quad$ need $c = 4$ conditions equation

$r = 4$

$\Rightarrow c = 4$

$\hat{l}_4 + \hat{l}_5 = 360°$, $\quad l_4 + v_4 + l_5 + v_5 = 360$, $\quad v_4 + v_5 = 360 - l_4 - l_5$

$\hat{l}_1 + \hat{l}_2 = \hat{l}_5$, $\quad l_1 + v_1 + l_2 + v_2 = l_5 + v_5$, $\quad v_1 + v_2 - v_5 = -l_1 - l_2 + l_5$

$\hat{l}_6 + \hat{l}_1 = 360$, $\quad l_6 + v_6 + l_1 + v_1 = 360$, $\quad v_6 + v_1 = 360 - l_6 - l_1$

$\hat{l}_3 = \hat{l}_7$, $\quad l_3 + v_3 = l_7 + v_7$, $\quad v_3 - v_7 = -l_3 + l_7$

- $v_4 + v_5 = 360 - 166 - 203$, $\quad v_4 + v_5 + 9 = 0$
- $v_1 + v_2 - v_5 = -120 - 82 + 203$, $\quad v_1 + v_2 - v_5 - 1 = 0$
- $v_6 + v_1 = 360 - 238 - 120$, $\quad v_6 + v_1 - 2 = 0$
- $v_3 - v_7 = -40 + 41$, $\quad v_3 - v_7 - 1 = 0$

$$\Phi' = v_1^2 + v_2^2 + v_3^2 + v_4^2 + v_5^2 + v_6^2 + v_7^2 - 2k_1(v_4+v_5+9) - 2k_2(v_1+v_2-v_5-1)$$
$$- 2k_3(v_6+v_1-2) - 2k_4(v_3-v_7-1)$$

$$\left.\begin{aligned}
\partial\phi'/\partial v_1 &= \cancel{2}v_1 - \cancel{2}k_2 - \cancel{2}k_3 = 0 \\
\partial\phi'/\partial v_2 &= \cancel{2}v_2 - \cancel{2}k_2 = 0 \\
\partial\phi'/\partial v_3 &= \cancel{2}v_3 - \cancel{2}k_4 = 0 \\
\partial\phi'/\partial v_4 &= \cancel{2}v_4 - \cancel{2}k_1 = 0 \\
\partial\phi'/\partial v_5 &= \cancel{2}v_5 - \cancel{2}k_1 + \cancel{2}k_2 = 0 \\
\partial\phi'/\partial v_6 &= \cancel{2}v_6 - \cancel{2}k_3 = 0 \\
\partial\phi'/\partial v_7 &= \cancel{2}v_7 + \cancel{2}k_4 = 0
\end{aligned}\right\} * \quad
\begin{aligned}
v_1 &= k_2 + k_3 \\
v_2 &= k_2 \\
v_3 &= k_4 \\
v_4 &= k_1 \\
v_5 &= k_1 - k_2 \\
v_6 &= k_3 \\
v_7 &= -k_4
\end{aligned}$$

$$\left.\begin{aligned}
\partial\phi'/\partial k_1 &= -\cancel{2}(v_4+v_5+9) = 0 \\
\partial\phi'/\partial k_2 &= -\cancel{2}(v_1+v_2-v_5-1) = 0 \\
\partial\phi'/\partial k_3 &= -\cancel{2}(v_6+v_1-2) = 0 \\
\partial\phi'/\partial k_4 &= -\cancel{2}(v_3-v_7-1) = 0
\end{aligned}\right\} \quad
\begin{aligned}
v_4 + v_5 &= -9 \\
v_1 + v_2 - v_5 &= 1 \\
v_6 + v_1 &= 2 \\
v_3 - v_7 &= 1
\end{aligned}$$

plug first 7 equations into last 4 (partitioned solution)

$$k_1 + k_1 - k_2 = -9$$
$$k_2 + k_3 + k_2 - k_1 + k_2 = 1$$
$$k_3 + k_2 + k_3 = 2$$
$$k_4 + k_4 = 1$$

$$2k_1 - k_2 = -9$$
$$-k_1 + 3k_2 + k_3 = 1$$
$$k_2 + 2k_3 = 2$$
$$2k_4 = 1$$

now solve for the k's

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 3 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} -9 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \quad \vec{K} = \begin{bmatrix} -5.625 \\ -2.25 \\ 2.125 \\ 0.5 \end{bmatrix}$$

plug these into * to obtain the v's

$$\begin{aligned}
v_1 &= -.125 \\
v_2 &= -2.25 \\
v_3 &= 0.5 \\
v_4 &= -5.625 \\
v_5 &= -3.375 \\
v_6 &= 2.125 \\
v_7 &= -0.5
\end{aligned}
\qquad
\hat{\ell} = \ell + v = \begin{bmatrix} 119.875 \\ 79.75 \\ 40.5 \\ 160.375 \\ 199.625 \\ 240.125 \\ 40.5 \end{bmatrix}$$

3. $n = 10$
$n_0 = 5$
$r = 5$

Observations only via substitution

need $c = r = 5$ condition equations

$\hat{l}_1 + \hat{l}_9 - \hat{l}_2 = 0$	$l_1 + V_1 + l_9 + V_9 - l_2 - V_2 = 0$	$V_1 + V_9 - V_2 = -l_1 - l_9 + l_2$
$\hat{l}_2 - \hat{l}_8 - \hat{l}_3 = 0$	$l_2 + V_2 - l_8 - V_8 - l_3 - V_3 = 0$	$V_2 - V_8 - V_3 = -l_2 + l_8 + l_3$
$\hat{l}_3 - \hat{l}_7 - \hat{l}_4 = 0$	$l_3 + V_3 - l_7 - V_7 - l_4 - V_4 = 0$	$V_3 - V_7 - V_4 = -l_3 + l_7 + l_4$
$\hat{l}_4 + \hat{l}_6 - \hat{l}_5 = 0$	$l_4 + V_4 + l_6 + V_6 - l_5 - V_5 = 0$	$V_4 + V_6 - V_5 = -l_4 - l_6 + l_5$
$\hat{l}_9 - \hat{l}_8 + \hat{l}_{10} = 0$	$l_9 + V_9 - l_8 - V_8 + l_{10} + V_{10} = 0$	$V_9 - V_8 + V_{10} = -l_9 + l_8 - l_{10}$

$$\left.\begin{aligned}
V_1 + V_9 - V_2 &= -19.5 - 1.8 + 21.8 = 0.5 \\
V_2 - V_8 - V_3 &= -21.8 + 7.3 + 15.3 = 0.8 \\
V_3 - V_7 - V_4 &= -15.3 + 4.7 + 10.4 = -0.2 \\
V_4 + V_6 - V_5 &= -10.4 - 1.9 + 11.7 = -0.6 \\
V_9 - V_8 + V_{10} &= -1.8 + 7.3 - 5.1 = 0.4
\end{aligned}\right\}$$

ok pick $n_0 = 5$ V's to fix model = 1,2,3,4,5 – make substitutions to solve for remaining 6,7,8,9,10 in terms of 1,2,3,4,5

$$V_9 = -V_1 + V_2 + 0.5$$
$$V_8 = V_2 - V_3 - 0.8$$
$$V_7 = V_3 - V_4 + 0.2$$
$$V_6 = -V_4 + V_5 - 0.6$$
$$\begin{aligned}
V_{10} &= -V_9 + V_8 + 0.4 \\
&= V_1 - V_2 - 0.5 + V_2 - V_3 - 0.8 + 0.4 \\
&= V_1 - V_3 - 0.9
\end{aligned}$$

*

$$\phi = V_1^2 + V_2^2 + V_3^2 + V_4^2 + V_5^2 + V_6^2 + V_7^2 + V_8^2 + V_9^2 + V_{10}^2$$
$$\phi = V_1^2 + V_2^2 + V_3^2 + V_4^2 + V_5^2 + (-V_4 + V_5 - 0.6)^2 + (V_3 - V_4 + 0.2)^2 + (V_2 - V_3 - 0.8)^2 + (-V_1 + V_2 + 0.5)^2 + (V_1 - V_3 - 0.9)^2$$

$$\partial\phi/\partial V_1 = \cancel{2}V_1 + \cancel{2}(-V_1 + V_2 + 0.5)(-1) + \cancel{2}(V_1 - V_3 - 0.9) = 0$$
$$\partial\phi/\partial V_2 = \cancel{2}V_2 + \cancel{2}(V_2 - V_3 - 0.8) + \cancel{2}(-V_1 + V_2 + 0.5) = 0$$
$$\partial\phi/\partial V_3 = \cancel{2}V_3 + \cancel{2}(V_3 - V_4 + 0.2) + \cancel{2}(V_2 - V_3 - 0.8)(-1) + \cancel{2}(V_1 - V_3 - 0.9)(-1) = 0$$
$$\partial\phi/\partial V_4 = \cancel{2}V_4 + \cancel{2}(-V_4 + V_5 - 0.6)(-1) + \cancel{2}(V_3 - V_4 + 0.2)(-1) = 0$$
$$\partial\phi/\partial V_5 = \cancel{2}V_5 + \cancel{2}(-V_4 + V_5 - 0.6) = 0$$

$$3V_1 - V_2 - V_3 = 0.5 + 0.9 = 1.4$$

$$-V_1 + 3V_2 - V_3 = 0.8 - 0.5 = 0.3$$

$$-V_1 - V_2 + 4V_3 - V_4 = -0.2 - 0.8 - 0.9 = -1.9$$

$$-V_3 + 3V_4 - V_5 = -0.6 + 0.2 = -0.4$$

$$-V_4 + 2V_5 = 0.6$$

solve by Matlab

$$\begin{bmatrix} 3 & -1 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 \\ -1 & -1 & 4 & -1 & 0 \\ 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{bmatrix} = \begin{bmatrix} 1.4 \\ 0.3 \\ -1.9 \\ -0.4 \\ 0.6 \end{bmatrix}, \quad \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{bmatrix} = \begin{bmatrix} 0.3529 \\ 0.0779 \\ -0.4192 \\ -0.2077 \\ 0.1962 \end{bmatrix}$$

now plug into *

$V_6 = -0.1962$

$V_7 = -0.0115$

$V_8 = -0.3029$

$V_9 = 0.225$

$V_{10} = -0.1279$

$\hat{\ell} = [19.8529;\ 21.8779;\ 14.8808;\ 10.1923;\ 11.8962;\ 1.7038;\ 4.6885;\ 6.9971;\ 2.025;\ 4.9721]$

---

4. $n = 4$, $n_0 = 1$, $r = 3$

write condition equations for Observations Only

need $c = r = 3$ equations with only: observations & numerical constants — no new variables:

$$\frac{d_1}{t_1} = \frac{d_2}{t_2}$$

$$\frac{d_1}{t_1} = \frac{d_3}{t_3}$$

$$\frac{d_1}{t_1} = \frac{d_4}{t_4}$$

note: by saying velocity is constant, that does not mean that we know what its value is, only that it does not change. some may have found that distinction confusing.