CE 597 Adj GeoSpa.Obs   HW2 solution

1. $z = a_0 + a_1 x + a_2 y$

$v_i - a_0 - a_1 x - a_2 y = -z_i$

$n = 9$

$n_0 = 3$

$r = 6$

$\sigma_0^2 = (0.2)^2$

$w_i = \frac{\sigma_0^2}{\sigma_i^2}$

	x	y	z
$\sigma_z = 0.1$	1	1	2.80
	1	2	3.18
	1	3	3.02
	2	1	3.34
	2	2	3.53
	2	3	3.62
$\sigma_z = 0.2$	3	1	3.66
	3	2	4.07
	3	3	4.97

$$B = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -2 \\ -1 & -1 & -3 \\ -1 & -2 & -1 \\ -1 & -2 & -2 \\ -1 & -2 & -3 \\ -1 & -3 & -1 \\ -1 & -3 & -2 \\ -1 & -3 & -3 \end{bmatrix}, \quad \Delta = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}, \quad f = -l = \begin{bmatrix} -2.80 \\ -3.18 \\ -3.02 \\ -3.34 \\ -3.53 \\ -3.62 \\ -3.66 \\ -4.07 \\ -4.97 \end{bmatrix}$$

$$W = \begin{bmatrix} 4 & & & & & & & & \\ & 4 & & & & \phi & & & \\ & & 4 & & & & & & \\ & & & 4 & & & & & \\ & & & & 4 & & & & \\ & & & & & 4 & & & \\ & \phi & & & & & 1 & & \\ & & & & & & & 1 & \\ & & & & & & & & 1 \end{bmatrix}$$

from matlab

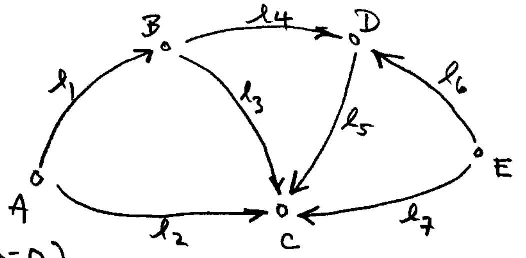
$$\Delta = \begin{bmatrix} 2.0289 \\ 0.5767 \\ 0.1839 \end{bmatrix}, \quad v = \begin{bmatrix} -.0106 \\ -.2067 \\ .1372 \\ .0261 \\ .0200 \\ .1139 \\ .2828 \\ .0567 \\ -.6594 \end{bmatrix}, \quad \hat{z} = \begin{bmatrix} 2.7894 \\ 2.9733 \\ 3.1572 \\ 3.3661 \\ 3.5500 \\ 3.7339 \\ 3.9428 \\ 4.1267 \\ 4.3106 \end{bmatrix}$$

```
% hw2_1_sol.m  20-sep-2011
% plane fit problem

B=[-1 -1 -1;-1 -1 -2;-1 -1 -3;-1 -2 -1;-1 -2 -2;
-1 -2 -3;-1 -3 -1;-1 -3 -2;-1 -3 -3];
l=[2.80;3.18;3.02;3.34;3.53;3.62;3.66;4.07;4.97];
f=-l;
% sig(1 -> 6) = 0.1
% sig(7 -> 9) = 0.2
% sig_0_sqr=(0.2)^2

W=diag([4 4 4 4 4 4 1 1 1]);
del=inv(B'*W*B)*B'*W*f
v=f - B*del
lhat=l + v
```

```
del =
    2.0289
    0.5767
    0.1839
v =
   -0.0106
   -0.2067
    0.1372
    0.0261
    0.0200
    0.1139
    0.2828
    0.0567
   -0.6594
lhat =
    2.7894
    2.9733
    3.1572
    3.3661
    3.5500
    3.7339
    3.9428
    4.1267
    4.3106
```

2. Level Network (Indirect Observations)

(a)



(A=0)
Datum Definition

$n = 7$
$n_0 = 4$
$r = 3$

$u = n_0 = 4$: B, C, D, E
$A = 0$
$\sigma_0^2 = (0.3)^2$

ℓ	σ
10.28	0.1
21.87	0.1
12.30	0.1
5.07	0.1
6.98	0.3
2.21	0.3
8.94	0.3

$\hat{\ell}_1 = B \qquad v_1 - B = -\ell_1$

$\hat{\ell}_2 = C \qquad v_2 - C = -\ell_2$

$\hat{\ell}_3 = C - B \qquad v_3 - C + B = -\ell_3$

$\hat{\ell}_4 = D - B \qquad v_4 - D + B = -\ell_4$

$\hat{\ell}_5 = C - D \qquad v_5 - C + D = -\ell_5$

$\hat{\ell}_6 = D - E \qquad v_6 - D + E = -\ell_6$

$\hat{\ell}_7 = C - E \qquad v_7 - C + E = -\ell_7$

$$v + \underset{B}{\left[\begin{array}{cccc} B & C & D & E \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 \end{array}\right]} \underset{\Delta}{\begin{bmatrix} B \\ C \\ D \\ E \end{bmatrix}} = \underset{f}{\begin{bmatrix} -10.28 \\ -21.87 \\ -12.30 \\ -5.07 \\ -6.98 \\ -2.21 \\ -8.94 \end{bmatrix}}, \quad W = \begin{bmatrix} 9 & & & & & & \emptyset \\ & 9 & & & & & \\ & & 9 & & & & \\ & & & 9 & & & \\ & & & & 1 & & \\ & & & & & 1 & \\ \emptyset & & & & & & 1 \end{bmatrix}$$

solve by matlab :

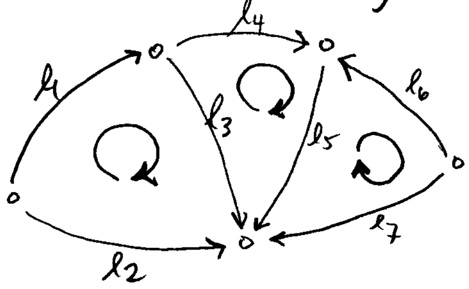
$$\Delta = \begin{bmatrix} 10.0475 \\ 22.1025 \\ 15.1301 \\ 13.0413 \end{bmatrix}, \quad v = \begin{bmatrix} -.2325 \\ .2325 \\ -.2451 \\ .0126 \\ -.0077 \\ -.1212 \\ .1212 \end{bmatrix}, \quad \hat{\ell} = \begin{bmatrix} 10.0475 \\ 22.1025 \\ 12.0549 \\ 5.0826 \\ 6.9723 \\ 2.0888 \\ 9.0612 \end{bmatrix}$$

(see listing)

(b) Observations Only : Same data, same σ's, same $n=7$, $n_0=4$, $r=3$, same W

So we need $c = r = 3$ condition equations

Sum around inner loops



$\hat{\ell}_1 + \hat{\ell}_3 - \hat{\ell}_2 = 0$

$\hat{\ell}_4 + \hat{\ell}_5 - \hat{\ell}_3 = 0$

$\hat{\ell}_7 - \hat{\ell}_5 - \hat{\ell}_6 = 0$

$$A = \left[\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 \end{array}\right]$$

$$Av = \underbrace{-A\ell}_{f}, \qquad Av = f$$

$$K = \begin{bmatrix} -2.0922 \\ .1135 \\ .1212 \end{bmatrix}, \quad v = \begin{bmatrix} -.2325 \\ .2325 \\ -.2451 \\ .0126 \\ -.0077 \\ -.1212 \\ .1212 \end{bmatrix}, \quad \hat{\ell} = \begin{bmatrix} 10.0475 \\ 22.1025 \\ 12.0549 \\ 5.0826 \\ 6.9723 \\ 2.0888 \\ 9.0612 \end{bmatrix}$$

```
% hw2_2_sol.m  20-sep-2011
% level net

l=[10.28;21.87;12.30;5.07;6.98;2.21;8.94];
sig_0_sqr=(03)^2;
% sig(1-4) = 0.1
% sig(5-7) = 0.3
W=diag([9 9 9 9 1 1 1]);

disp('first solve by indirect observations');

B=[-1 0 0 0;
0 -1 0 0;
1 -1 0 0;
1 0 -1 0;
0 -1 1 0;
0 0 -1 1;
0 -1 0 1];
f=-l;

del=inv(B'*W*B)*B'*W*f
v=f-B*del
lhat=l+v

disp('now solve by observations only');

A=[1 -1 1 0 0 0 0;0 0 -1 1 1 0 0;0 0 0 0 -1 -1 1];
f=-A*l;
Q=inv(W);
k=inv(A*Q*A')*f
v=Q*A'*k
lhat=l+v
```

```
first solve by indirect observations
del =
   10.0475
   22.1025
   15.1301
   13.0413
v =
   -0.2325
    0.2325
   -0.2451
    0.0126
   -0.0077
   -0.1212
    0.1212
lhat =
   10.0475
   22.1025
   12.0549
    5.0826
    6.9723
    2.0888
    9.0612
now solve by observations only
k =
   -2.0922
    0.1135
    0.1212
v =
   -0.2325
    0.2325
   -0.2451
    0.0126
   -0.0077
   -0.1212
    0.1212
lhat =
   10.0475
   22.1025
   12.0549
    5.0826
    6.9723
    2.0888
    9.0612
diary off
```

3. what do the "observations only" equations look like for the plane fit problem? Solve it 4 ways:

(a) by <u>elimination of parameters</u> (show for first* 4 points, will be same for points 5, 6, 7, 8, 9)

$$z_1 = a_0 + a_1 x_1 + a_2 y_1 \longrightarrow a_0 = z_1 - a_1 x_1 - a_2 y_1$$, plug into remaining 3 eqn's

$$z_2 = a_0 + a_1 x_2 + a_2 y_2$$

$$z_3 = a_0 + a_1 x_3 + a_2 y_3$$

$$z_4 = a_0 + a_1 x_4 + a_2 y_4$$

$\ast$ $\boxed{\text{Careful about order of points and elimination!}}$

$$z_2 = z_1 - a_1 x_1 - a_2 y_1 + a_1 x_2 + a_2 y_2 = z_1 + (x_2 - x_1) a_1 + (y_2 - y_1) a_2$$

$$z_3 = z_1 - a_1 x_1 - a_2 y_1 + a_1 x_3 + a_2 y_3 = z_1 + (x_3 - x_1) a_1 + (y_3 - y_1) a_2$$

$$z_4 = z_1 - a_1 x_1 - a_2 y_1 + a_1 x_4 + a_2 y_4 = z_1 + (x_4 - x_1) a_1 + (y_4 - y_1) a_2$$

$\rightarrow$ solve for a_1

$$a_1 = \frac{(z_2 - z_1) - (y_2 - y_1) a_2}{x_2 - x_1} = \frac{z_2 - z_1}{x_2 - x_1} - \frac{y_2 - y_1}{x_2 - x_1} a_2$$, plug into remaing 2 eqn's

$$z_3 = z_1 + \frac{(x_3 - x_1)(z_2 - z_1)}{x_2 - x_1} - \frac{(x_3 - x_1)(y_2 - y_1)}{x_2 - x_1} a_2 + (y_3 - y_1) a_2$$

$$z_4 = z_1 + \frac{(x_4 - x_1)(z_2 - z_1)}{x_2 - x_1} - \frac{(x_4 - x_1)(y_2 - y_1)}{x_2 - x_1} a_2 + (y_4 - y_1) a_2$$

$$z_3 = z_1 + \frac{(x_3 - x_1)(z_2 - z_1)}{x_2 - x_1} + \left[(y_3 - y_1) - \frac{(x_3 - x_1)(y_2 - y_1)}{x_2 - x_1}\right] a_2$$

$$z_4 = z_1 + \frac{(x_4 - x_1)(z_2 - z_1)}{x_2 - x_1} + \left[(y_4 - y_1) - \frac{(x_4 - x_1)(y_2 - y_1)}{x_2 - x_1}\right] a_2$$

$$\rightarrow a_2 = \frac{(z_3 - z_1) - \frac{(x_3 - x_1)(z_2 - z_1)}{x_2 - x_1}}{(y_3 - y_1) - \frac{(x_3 - x_1)(y_2 - y_1)}{x_2 - x_1}} = \frac{(x_2 - x_1)(z_3 - z_1) - (x_3 - x_1)(z_2 - z_1)}{(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)},$$

plug this into last equation for z_4

$$z_4 = z_1 + \frac{(x_4 - x_1)(z_2 - z_1)}{x_2 - x_1} + \left[(y_4 - y_1) - \frac{(x_4 - x_1)(y_2 - y_1)}{(x_2 - x_1)}\right] \frac{(x_2 - x_1)(z_3 - z_1) - (x_3 - x_1)(z_2 - z_1)}{(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)}$$

rearrange:

$$\frac{(z_4 - z_1) - \frac{(x_4 - x_1)(z_2 - z_1)}{(x_2 - x_1)}}{(y_4 - y_1) - \frac{(x_4 - x_1)(y_2 - y_1)}{(x_2 - x_1)}} = \frac{(x_2 - x_1)(z_3 - z_1) - (x_3 - x_1)(z_2 - z_1)}{(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)}$$

rearrange on left:

$$\boxed{\frac{(X_2-x_1)(z_4-z_1)-(x_4-x_1)(z_2-z_1)}{(x_2-x_1)(y_4-y_1)-(x_4-x_1)(y_2-y_1)}=\frac{(x_2-x_1)(z_3-z_1)-(x_3-x_1)(z_2-z_1)}{(x_2-x_1)(y_3-y_1)-(x_3-x_1)(y_2-y_1)}}$$

that's it. that is the equation for point #4. Just replace X_4, Y_4, Z_4 with X_5, Y_5, Z_5 for the next point, etc.

(b) second approach:
we know that the equation of a plane through 3 points is:

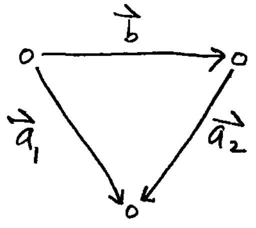
$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0 ,$$

The condition equation for 4 points in a plane is

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

rewrite this for point 5, 6, 7, etc.

(c) third approach: for those that have studied photogrammetry we have the <u>coplanarity</u> condition equation for 3 vectors:

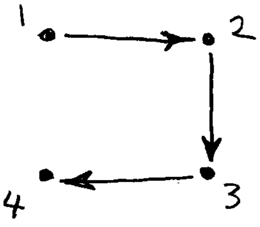


$$\vec{b}\cdot(\vec{a}_1\times\vec{a}_2) = 0$$

OR

$$\begin{vmatrix} b_x & b_y & b_z \\ a_{1x} & a_{1y} & a_{1z} \\ a_{2x} & a_{2y} & a_{2z} \end{vmatrix} = 0$$

To write an equation that forces 4 points to lie in a plane we just make coplanarity for these 3 vectors



then write the same equation for
1, 2, 3, 5
1, 2, 3, 6
1, 2, 3, 7 etc.

$$\begin{vmatrix} X_2-X_1 & Y_2-Y_1 & Z_2-Z_1 \\ X_3-X_2 & Y_3-Y_2 & Z_3-Z_2 \\ X_4-X_3 & Y_4-Y_3 & Z_4-Z_3 \end{vmatrix} = 0$$

(d) elimination procedure — but eliminate all parameters at once. Show elimination with 3 equations and substitute into number 4. Can repeat substitution into 5, 6, 7, … . Again, the order is significant, the first 3 points <u>must</u> define a plane (if they fall along a line it will not work, gridded data can present challenges)

$$\left.\begin{aligned} z_1 &= a_0 + a_1 x_1 + a_2 y_1 \\ z_2 &= a_0 + a_1 x_2 + a_2 y_2 \\ z_3 &= a_0 + a_1 x_3 + a_2 y_3 \end{aligned}\right\} \qquad \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

$$z_4 = a_0 + a_1 x_4 + a_2 y_4$$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

Substitute the expressions for a_0, a_1, a_2 into equations for $z_4, z_5, z_6, \ldots$

Any LS problem can be solved by indirect observations (parameters) or by observations only. This exercise just shows how to convert condition equations from I/O to O/O. Naturally you usually choose the method that is <u>easiest</u> and <u>most</u> <u>convenient</u>.