Differentiation with respect to vector 6-1

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\frac{d}{d\vec{x}}(a_1x_1 + a_2x_2) = \left[\frac{d}{dx_1}(\cdot) \quad \frac{d}{dx_2}(\cdot)\right]$

my convention $\frac{d}{d\vec{x}}(\cdot)$: row vector ←

others choose $\frac{d}{d\vec{x}}(\cdot)$: column vector $\boxed{\frac{d}{d\vec{x}}Ax = A}$

$\frac{d}{d\vec{x}}(a_1x_1 + a_2x_2) = [a_1 \quad a_2]$

$\frac{d}{dx}[a_1 \quad a_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (a_1 \quad a_2)$

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$\frac{d}{d\vec{x}}Ax = \frac{d}{dx}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{matrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{matrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ 6-2

$Ax = b$, $\frac{d}{d\vec{x}}(Ax) = \frac{d}{d\vec{x}}b = \begin{bmatrix} \frac{\partial}{\partial x_1}b_1 & \frac{\partial}{\partial x_2}b_1 \\ \frac{\partial}{\partial x_1}b_2 & \frac{\partial}{\partial x_2}b_2 \end{bmatrix}$

$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $\boxed{\frac{d}{d\vec{x}}Ax = A}$

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not so obvious $\frac{d}{d\vec{x}} \underbrace{x^T A x}$ = ? A symmetric 63

quadratic form

$$[x_1 \ \ x_2]\begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = [a_{11}x_1 + a_{12}x_2 \quad a_{12}x_1 + a_{22}x_2]\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\frac{d}{dx}\left[a_{11}x_1^2 + a_{12}x_1x_2 + a_{12}x_1x_2 + a_{22}x_2^2\right] = ? = \left[2a_{11}x_1 + \underline{a_{12}x_2 + a_{12}x_2} \quad \underline{a_{12}x_1 + a_{12}x_1} + 2a_{22}x_2\right]$$

$$[2a_{11}x_1 + 2a_{12}x_2 \quad 2a_{12}x_1 + 2a_{22}x_2]$$

$$2[a_{11}x_1 + a_{12}x_2 \quad a_{12}x_1 + a_{22}x_2]$$

$$2[x_1 \ \ x_2]\begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} = 2x^T A$$

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$$\boxed{\frac{d}{d\vec{x}} x^T A x = 2x^T A}$$

6-4

A: sym

$x^T A y$: bilinear form = scalar

$$\underbrace{\underbrace{(1,2)(2,2)}\underbrace{(2,1)}}_{(1,1)}$$

$x^T A y = y^T A^T x$

because of

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$\Phi = v^T W v$, $v + B\delta = f$ $f = d - \ell$ 6-5

resid. coef mx. par. vec. right side

$v = f - B\delta$

W = weight matrix always sym. !

$\Phi = (f - B\delta)^T W (f - B\delta)$

$\Phi = f^T W f + \delta^T B^T W B \delta - f^T W B \delta - \underline{\delta^T B^T W f}$

$- f^T W B \delta$

$\Phi = f^T W f + \delta^T B^T W B \delta - 2 f^T W B \delta$

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$\Phi = f^T W f + \delta^T \underbrace{B^T W B}\delta - 2 f^T W B \delta$ minimize 6-6

$\frac{d\Phi}{\partial\delta} = \cancel{2}\underbrace{\delta^T B^T W B}_{\substack{(1,u)(u,n)(n,n)(n,u) \\ (1,u)}} - \cancel{2} \underset{(1,u)}{f^T W B} = \underset{(1,u)}{0}$

n : # obs
u : # par
c : # C.E.

$\underset{(u,1)}{B^T W B \delta} - \underset{(u,1)}{B^T W f} = \underset{(u,1)}{0}$

normal equations vector/matrix form

$$\boxed{B^T W B \delta = B^T W f}$$

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$$\underbrace{B^T W B}\, \Delta = \underbrace{B^T W f}$$

$$N \Delta = t$$

$$\boxed{\Delta = N^{-1} t}$$

$$v = f - B\Delta$$

$$l + v = \hat{l}$$

to solve LS problem: analyze problem, W write C.E. $v + B\Delta = f$

$$\boxed{\begin{aligned} \Delta &= (B^T W B)^{-1} B^T W f \\ v &= f - B\Delta \\ \hat{l} &= l + v \end{aligned}}$$

6-7

Indirect Observation problem

$v + B\Delta = f$

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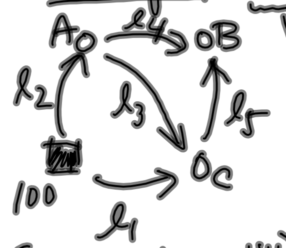
Example $\underbrace{I/O}$



(arrow points up hill)

$n = 5$

$n_0 = 3$

$r = 2$

$u = n_0 = 3$

choice of par: A, B, C

$\hat{l}_1 = C - 100$, $v_1 - C = -100 - l_1$

$\hat{l}_2 = A - 100$, $v_2 - A = -100 - l_2$

$\hat{l}_3 = C - A$, $v_3 - C + A = -l_3$

$\hat{l}_4 = B - A$, $v_4 - B + A = -l_4$

$\hat{l}_5 = B - C$, $v_5 - B + C = -l_5$

6-8

$v + B\Delta = f$

$c = n = 5$ cond. eqn.

$$\vec{l} = \begin{bmatrix} 10 \\ 8 \\ 3 \\ 4 \\ 1 \end{bmatrix} \qquad \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix}}_{v} + \underbrace{\begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} A \\ B \\ C \end{bmatrix}}_{\Delta} = \underbrace{\begin{bmatrix} -100 \\ -100 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{d} - \underbrace{\begin{bmatrix} l_1 \\ l_2 \\ l_3 \\ l_4 \\ l_5 \end{bmatrix}}_{l}$$

$$\underbrace{d - l}_{f}$$

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obs : equal precision + uncorrelated 6-9

$\underline{W=I}$

$$\Delta = (B^T W B)^{-1} B^T W f \rightarrow \Delta = \begin{bmatrix} 107.625 \\ 111.500 \\ 110.375 \end{bmatrix}$$

$$v = f - B\Delta = \underbrace{\begin{bmatrix} .375 \\ -.375 \\ -.25 \\ -.125 \\ .125 \end{bmatrix}}_{\text{☺☺}}, \quad \hat{l} = l + v = \begin{bmatrix} 10.375 \\ 7.625 \\ 2.750 \\ 3.875 \\ 1.125 \end{bmatrix}$$

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observation only 6-10

$w_i \sim \frac{1}{\sigma_i^2}$ $\quad w_i = \frac{k}{\sigma_i^2}$ $\quad k = \sigma_0^2$ $\quad w_i = \frac{\sigma_0^2}{\sigma_i^2}$

σ_0^2 : you choose value
- variance of unit weight
- reference variance
- a priori reference var.
- pre-adjustment r.v.
- prior

$\hat{\sigma}_0^2$ post adjustment estimate of ref. var.
a posteriori ref. var.

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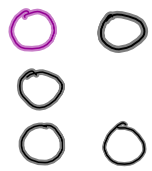
6-11

$$W = \begin{bmatrix} w_1 & & & & \\ & w_2 & & & \\ & & w_3 & & \\ & & & \ddots & \\ & & & & w_n \end{bmatrix} = \begin{bmatrix} \sigma_0^2/\sigma_1^2 & & & \\ & \sigma_0^2/\sigma_2^2 & & \\ & & \ddots & \\ & & & \sigma_0^2/\sigma_n^2 \end{bmatrix}$$

$$W = \sigma_0^2 \begin{bmatrix} 1/\sigma_1^2 & & & \\ & 1/\sigma_2^2 & & \\ & & \ddots & \\ & & & 1/\sigma_n^2 \end{bmatrix}$$

$$W = \sigma_0^2 \Sigma^{-1}$$



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