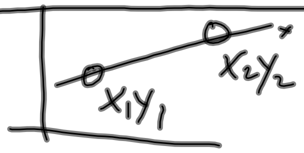
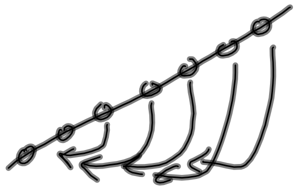
25-1

a homogeneous linear system (square) has a nontrivial solution iff determinant = 0



$$c_1x + c_2y + c_3 = 0$$
$$c_1x_1 + c_2y_1 + c_3 = 0$$
$$c_1x_2 + c_2y_2 + c_3 = 0$$

$$\begin{bmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

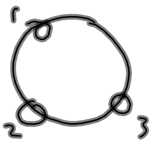


$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

Nov 14-10:27 AM

25-2

Circle through 3 points



$$(x-x_c)^2 + (y-y_c)^2 - R = 0$$
$$x^2 + x_c^2 - 2xx_c + y^2 + y_c^2 - 2yy_c - R = 0$$
$$x^2 + y^2 - 2xx_c - 2yy_c + \underbrace{x_c^2 + y_c^2 - R} = 0$$

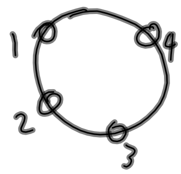
$$\begin{cases} c_1(x^2+y^2) + c_2x + c_3y + c_4 = 0 \leftarrow \\ c_1(x_1^2+y_1^2) + c_2x_1 + c_3y_1 + c_4 = 0 \\ c_1(x_2^2+y_2^2) + c_2x_2 + c_3y_2 + c_4 = 0 \\ c_1(x_3^2+y_3^2) + c_2x_3 + c_3y_3 + c_4 = 0 \end{cases}$$

Nov 14-10:28 AM

25-3

$$\begin{bmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\det([\cdot]) = 0$$



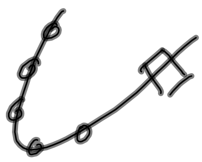
Nov 14-10:28 AM

25-4

general conic section through 5 pts.

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

$$c_1 \quad c_2 \quad \cdots \quad c_6$$

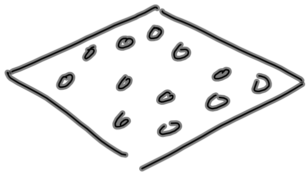


$$\begin{vmatrix} x^2 & xy & y^2 & x & y & 1 \\ x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ & \vdots & & & & \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 & 1 \end{vmatrix} = 0$$

Nov 14-10:28 AM

plane through 3 points

25-5



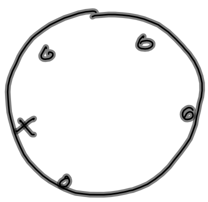
$$c_1 x + c_2 y + c_3 z + c_4 = 0$$

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

Nov 14-10:28 AM

Sphere through 4 points

25-6



$$(x-x_c)^2 + (y-y_c)^2 + (z-z_c)^2 - R = 0$$

$$\vdots$$

$$c_1(x^2+y^2+z^2) + c_2 x + c_3 y + c_4 z + c_5 = 0$$

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ x_1^2+y_1^2+z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2+y_2^2+z_2^2 & x_2 & y_2 & z_2 & 1 \\ \vdots & & & & \\ x_4^2+y_4^2+z_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

Nov 14-10:28 AM

Solve LSE : LS with equality constraints 25-7

GLS = $C = r + u$ unknowns all independent

LSE : u total # parameters

u' independent

$u - u'$ dependent $s = u - u'$, $u' = u - s$

$C = r + u'$

$C = r + u - s$

$$\boxed{\underbrace{C}_{\text{\# cond. eqn}} + \underbrace{s}_{\text{\# constr. eqn}} = \underbrace{r}_{\text{redundancy}} + \underbrace{u}_{\text{total params}}}$$

Nov 14-10:28 AM

cond. eqn : $\underset{c,n}{A}\,\underset{n,1}{v} + \underset{c,u}{B}\,\underset{u,1}{\Delta} = \underset{c,1}{f}$ 25-8

constr. eqn : $\underset{s,u}{C}\,\underset{u,1}{\Delta} = \underset{s,1}{g}$

<u>First solution by elimination</u>

Nov 14-10:28 AM

$$Av + B_1 \Delta_1 + B_2 \Delta_2 = f \qquad B_\Delta : B\begin{pmatrix}\Delta_1\\ \Delta_2\end{pmatrix} \quad \text{25-9}$$

$$\underbrace{C_1}\Delta_1 + C_2 \Delta_2 = g \qquad (B_1\, B_2)\begin{pmatrix}\Delta_1\\ \Delta_2\end{pmatrix}$$

arrange so C_1 is square + nonsingular
full rank

$$\Delta_1 = C_1^{-1}(g - C_2\Delta_2)$$

$$Av + B_1C_1^{-1}(g - C_2\Delta_2) + B_2\Delta_2 = f$$

$$Av - B_1C_1^{-1}C_2\Delta_2 + B_2\Delta_2 = f - B_1C_1^{-1}g$$

$$Av + \underbrace{(B_2 - B_1C_1^{-1}C_2)}_{\bar{B}}\Delta_2 = \underbrace{f - B_1C_1^{-1}g}_{\bar{f}}$$

Nov 14-10:28 AM

$$\boxed{Av + \bar{B}\Delta_2 = \bar{f}}$$ use GCS to solve this unconstrained problem 25-10

when done $\Delta_2, \lambda, v, \ldots$

$$\Delta_1 = C_1^{-1}(g - C_2\Delta_2)$$

Nov 14-10:29 AM

$$Av + B\delta = f \qquad \phi = v^T W v \qquad 25\text{-}11$$

$$C\Delta = g \qquad (\text{If } A = I)$$

$$\phi' = v^T W v - 2k^T(Av + B\delta - f) - 2k_c^T(C\delta - g)$$

$$\left\{ \begin{array}{l} -2(Av + B\delta - f)^T k \\ -2(C\delta - g)^T k_c \end{array} \right.$$

$$\frac{\partial \phi'}{\partial v} = \cancel{2}v^T W - \cancel{2}k^T A = 0$$

$$\frac{\partial \phi'}{\partial \Delta} = -\cancel{2}k^T B - \cancel{2}k_c^T C = 0$$

$$\frac{\partial \phi'}{\partial k} = -\cancel{2}(Av + B\delta - f)^T = 0$$

$$\partial \phi' / \partial k_c = -\cancel{2}(C\delta - g)^T = 0$$

Nov 14-10:29 AM

25-12

transpose:

$$\left. \begin{array}{l} Wv - A^T k = 0 \\ -B^T k - C^T k_c = 0 \\ -(Av + B\delta - f) = 0 \\ -(C\delta - g) = 0 \end{array} \right\} \begin{array}{l} -Wv + A^T k = 0 \\ B^T k + C^T k_c = 0 \\ Av + B\delta = f \\ C\delta = g \end{array}$$

$$\begin{bmatrix} -W & 0 & A^T & 0 \\ 0 & 0 & B^T & C^T \\ A & B & 0 & 0 \\ 0 & C & 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ \Delta \\ k \\ k_c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ f \\ g \end{bmatrix}$$

full normal equations

Nov 14-10:29 AM

$$\begin{bmatrix} W & A^T & 0 & 0 \\ A & 0 & B & 0 \\ 0 & B^T & 0 & C^T \\ 0 & 0 & C & 0 \end{bmatrix} \begin{bmatrix} v \\ k \\ \Delta \\ k_c \end{bmatrix} = \begin{bmatrix} 0 \\ f \\ 0 \\ g \end{bmatrix} \leftarrow \text{O.L.S. p. 215}$$

25-13

1, 3, 2, 4

(1) $-Wv + A^T k = 0$

(2) $B^T k + C^T k_c = 0$

(3) $Av + B\Delta = f$

(4) $C\Delta = g$

$Wv = A^T k, \quad v = QA^T k$

sub into (3)

$AQA^T k + B\Delta = f$

$k = (AQA^T)^{-1}(f - B\Delta)$

$k = W_e(f - B\Delta)$

sub into (2)

$B^T W_e (f - B\Delta) + C^T k_c = 0$

$\underbrace{-B^T W_e B}\Delta + C^T k_c = -\underbrace{B^T W_e f}$

$$\boxed{\begin{aligned} -N\Delta + C^T k_c &= -t \\ C\Delta &= g \end{aligned}} \leftarrow$$

Two possible ways to proceed:

25-14

1. Continue with elimination, N must be invertible

2. stop.

#1 $N\Delta = t + C^T k_c$

$\Delta = N^{-1} t + N^{-1} C^T k_c$

sub into 2nd eqn. $C[N^{-1} t + N^{-1} C^T k_c] = g$

25-15

$$CN^{-1}C^T k_c = g - CN^{-1}t$$

$$k_c = (CN^{-1}C^T)^{-1}(g - CN^{-1}t)$$

Nov 14-10:29 AM