J. Bethel, 4 Feb 2013

Obtaining Approximations for Exterior Orientation Using the 8-parameter Transformation

Note: The first part of this was taken from Appendix C of Photogrammetry by Moffitt & Mikhail, 1980, Harper+Row (out of print)

See Figure 1 for the definitions of the variables.

$$x' = u'\cos\alpha' + v'\sin\alpha' + d' \quad \text{(image)} \qquad [1]$$
$$y' = -u'\sin\alpha' + v'\cos\alpha' + e'$$

$$u = x\cos\alpha + y\sin\alpha + d \quad \text{(object)} \qquad [2]$$
$$v = -x\sin\alpha + y\cos\alpha + e$$

<u>Three</u> parameters relate $\binom{u'}{v'} \to \binom{x'}{y'}$ and <u>three</u> parameters relate $\binom{u}{v} \to \binom{x}{y}$

That leaves two parameters to relate $\binom{x}{y} \to \binom{x'}{y'}$

- u, v : observed coordinates in object system (plane)
- u', v' : observed coordinates in image system (plane)
- x, y : auxiliary system on object plane with origin at projection of the isocenter point.
- x', y' : auxiliary system on image plane with origin at the isocenter.

Scale in the y' direction at A_0' (actually reciprocal of the usual scale):

$$\frac{y}{y'} = K \quad \text{OR} \quad y = y'K \qquad [3]$$

As in figure 2, the scale K is also

$$K = \frac{IA_0}{BA_0'} = \frac{h}{f-g} = \frac{h}{f - x'\sin t} = \frac{h/\sin t}{f/\sin t - x'} \qquad [4]$$

from figure 3, (t is the tilt angle)

$$f = f' \sin t \quad \text{and} \quad h = h' \sin t, \quad \text{OR}$$

$$f' = f/\sin t \quad \text{and} \quad h' = h/\sin t \qquad [5]$$

so,

$$K = \frac{h'}{f' - x'} \qquad [6]$$

combine [3] & [6],

$$y = \frac{y' h'}{f' - x'} \qquad [7]$$

to obtain an analogous equation for x, see figure 2

$$x = K(\overline{BA_0'}) \qquad [8]$$

but, $BA_0'I'$ is an isosceles triangle where,

$$BA_0' = A_0'I = x' \qquad [9]$$

combining [8] & [9],

$$x = K x' \qquad [10]$$

combining this with [6]

$$x = \frac{h' x'}{f' - x'}$$

interesting conclusion: with origin at the isocenter, scale in x' and y' directions are <u>equal</u>!

now substitute [1] into [10] and [7]

$$x = \frac{h'(u'\cos\alpha' + v'\sin\alpha' + d')}{f' - (u'\cos\alpha' + v'\sin\alpha' + d')}$$

$$y = \frac{h'(-u'\sin\alpha' + v'\cos\alpha' + e')}{f' - (u'\cos\alpha' + v'\sin\alpha' + d')} \qquad [11]$$

subsitituting these into [2],

$$u = \frac{(u'\cos\alpha' + v'\sin\alpha' + d')\,h'\cos\alpha}{f' - (u'\cos\alpha' + v'\sin\alpha' + d')} + \frac{(-u'\sin\alpha' + v'\cos\alpha' + e')\,h'\sin\alpha}{f' - (u'\cos\alpha' + v'\sin\alpha' + d')} + d$$

$$v = \frac{-(u'\cos\alpha' + v'\sin\alpha' + d')\,h'\sin\alpha}{f' - (u'\cos\alpha' + v'\sin\alpha' + d')} + \frac{(-u'\sin\alpha' + v'\cos\alpha' + e')\,h'\cos\alpha}{f' - (u'\cos\alpha' + v'\sin\alpha' + d')} + e \qquad [12]$$

now combine terms into the classic 8-parameter form.
change the common denominator to

$$(f' - d') - (u'\cos\alpha' + v'\sin\alpha') \qquad [13]$$

then divide both numerator and denominator by $(f'-d')$,
the new denominator will be,

$$1 - \frac{(u'\cos\alpha' + v'\sin\alpha')}{(f'-d')} \qquad [14]$$

$$u = \frac{(u'\cos\alpha' + v'\sin\alpha' + d')\,h'\cos\alpha \,/\, (f'-d')}{1 - \frac{(u'\cos\alpha' + v'\sin\alpha')}{(f'-d')}} + \frac{(-u'\sin\alpha' + v'\cos\alpha' + e')\,h'\sin\alpha \,/\, (f'-d')}{1 - \frac{(u'\cos\alpha' + v'\sin\alpha')}{(f'-d')}} + \frac{d - d(u'\cos\alpha' + v'\sin\alpha') \,/\, (f'-d')}{1 - \frac{(u'\cos\alpha' + v'\sin\alpha')}{(f'-d')}}$$

$$v = \frac{-(u'\cos\alpha' + v'\sin\alpha' + d')\,h'\sin\alpha \,/\, (f'-d')}{1 - \frac{(u'\cos\alpha' + v'\sin\alpha')}{(f'-d')}} + \frac{(-u'\sin\alpha' + v'\cos\alpha' + e')\,h'\cos\alpha \,/\, (f'-d')}{1 - \frac{(u'\cos\alpha' + v'\sin\alpha')}{(f'-d')}} + \frac{e - e(u'\cos\alpha' + v'\sin\alpha') \,/\, (f'-d')}{1 - \frac{(u'\cos\alpha' + v'\sin\alpha')}{(f'-d')}} \qquad [15]$$

if you look at equation(s) [15] you see that they are in the form of the classical 8-parameter transformation form,

$$u = \frac{a_0 + a_1 u' + a_2 v'}{1 + c_1 u' + c_2 v'}$$

$$v = \frac{b_0 + b_1 u' + b_2 v'}{1 + c_1 u' + c_2 v'} \qquad [16]$$

by inspection we can write down the 8 coefficients $a_0, a_1, a_2, b_0, b_1, b_2, c_1, c_2$, in terms of the physically meaningful parameters $\alpha, d, e, \alpha', d', e', h', f'$.

$$\boxed{c_1 = \frac{-\cos\alpha'}{f'-d'}}$$

$$\boxed{c_2 = \frac{-\sin\alpha'}{f'-d'}} \qquad [17]$$

$$a_1 = [h'(\cos\alpha'\cos\alpha - \sin\alpha'\sin\alpha) - d\cos\alpha']/(f'-d')$$

recall trig identities:

$$\sin(a+b) = \sin a\cos b + \cos a \sin b$$
$$\sin(a-b) = \sin a\cos b - \cos a \sin b$$
$$\cos(a+b) = \cos a\cos b - \sin a \sin b$$
$$\cos(a-b) = \cos a\cos b + \sin a \sin b$$

$$\boxed{a_1 = [h'\cos(\alpha'+\alpha) - d\cos\alpha']/(f'-d')} \qquad [18]$$

$$a_2 = [h'(\sin\alpha'\cos\alpha + \cos\alpha'\sin\alpha) - d\sin\alpha']/(f'-d')$$

$$\boxed{a_2 = [h'\sin(\alpha'+\alpha) - d\sin\alpha']/(f'-d')} \qquad [19]$$

$$\boxed{a_0 = h'(d'\cos\alpha + e'\sin\alpha)/(f'-d') + d} \quad [20]$$

$$b_1 = \left[h'(-\cos\alpha'\sin\alpha - \sin\alpha'\cos\alpha) - e\cos\alpha'\right]/(f'-d')$$

$$\boxed{b_1 = -\left[h'\sin(\alpha'+\alpha) + e\cos\alpha'\right]/(f'-d')} \quad [21]$$

$$b_2 = \left[h'(\cos\alpha'\cos\alpha - \sin\alpha'\sin\alpha) - e\sin\alpha'\right]/(f'-d')$$

$$\boxed{b_2 = \left[h'(\cos(\alpha'+\alpha) - e\sin\alpha'\right]/(f'-d')} \quad [22]$$

$$\boxed{b_0 = h'(-d'\sin\alpha + e'\cos\alpha)/(f'-d') + e} \quad [23]$$

Now rearrange and combine to solve for the physical parameters, $\alpha, d, e, \alpha', d', e', f', h'$, given $a_0, a_1, a_2, b_0, b_1, b_2, c_1, c_2$.

$$c_1^2 + c_2^2 = \frac{\sin^2\alpha' + \cos^2\alpha'}{(f'-d')^2} = \frac{1}{(f'-d')^2}$$

$$\boxed{(f'-d') = \frac{1}{\sqrt{c_1^2 + c_2^2}}} \quad [24]$$

$$\boxed{\alpha' = \tan^{-1}\left[\frac{-c_2}{-c_1}\right]} \quad [25]$$

Subtract [22] from [18], and add [19] to [21]

$$a_1 - b_2 = (-d\cos\alpha' + e\sin\alpha')/(f'-d')$$

$$a_2 + b_1 = (-d\sin\alpha' - e\cos\alpha')/(f'-d')$$

rearrange these two equations into vector/matrix form

$$\begin{bmatrix}(f'-d')(a_1-b_2)\\(f'-d')(a_2+b_1)\end{bmatrix} = \begin{bmatrix}-\cos\alpha' & +\sin\alpha'\\ -\sin\alpha' & -\cos\alpha'\end{bmatrix}\begin{bmatrix}d\\e\end{bmatrix}$$

$$\boxed{\begin{bmatrix}d\\e\end{bmatrix} = \begin{bmatrix}-\cos\alpha' & +\sin\alpha'\\ -\sin\alpha' & -\cos\alpha'\end{bmatrix}^{-1}\begin{bmatrix}(f'-d')(a_1-b_2)\\(f'-d')(a_2+b_1)\end{bmatrix}} \qquad [26]$$

rearrange expressions for a_1 & b_1, [18] & [21]

$$a_1(f'-d') + d\cos\alpha' = h'\cos(\alpha'+\alpha)$$

$$b_1(f'-d') + e\cos\alpha' = -h'\sin(\alpha'+\alpha)$$

call the left sides S_1 and S_2 respectively,

$$S_1 = h'\cos(\alpha'+\alpha)$$

$$S_2 = -h'\sin(\alpha'+\alpha)$$

$$\boxed{h' = \sqrt{S_1^2+S_2^2}} \qquad [27]$$

$$\alpha'+\alpha = \tan^{-1}\left(\frac{-S_2}{S_1}\right)$$

use result from [25],

$$\boxed{\alpha = \tan^{-1}\left(\frac{-S_2}{S_1}\right) - \alpha'} \qquad [28]$$

rearrange expressions for a_0 & b_0, [20] & [23]

$$(a_0 - d)(f' - d')/h' = d'\cos\alpha + e'\sin\alpha$$

$$(b_0 - e)(f' - d')/h' = -d'\sin\alpha + e'\cos\alpha$$

put these two equations in vector/matrix form,

$$\begin{bmatrix} (a_0 - d)(f' - d')/h' \\ (b_0 - e)(f' - d')/h' \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} d' \\ e' \end{bmatrix}$$

$$\boxed{\begin{bmatrix} d' \\ e' \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}^{-1} \begin{bmatrix} (a_0 - d)(f' - d')/h' \\ (b_0 - e)(f' - d')/h' \end{bmatrix}} \qquad [29]$$

from [24] we know $(f' - d')$,

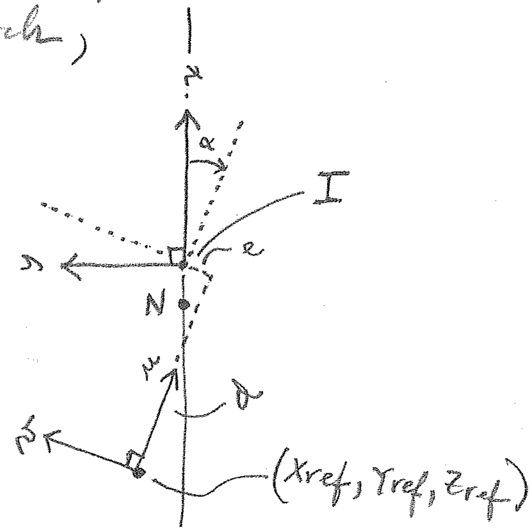
$$\boxed{f' = (f' - d') + d'} \qquad [30]$$

We have now solved for $\alpha, d, e, \alpha', d', e', f', h'$. These parameters do not completely fix the geometry in figure 1. The "hinge" between the image plane and the object plane can open and close. This corresponds to the parallelogram GSHP changing shape, with t, f, and h all changing. We need to fix <u>one</u> of t, f, h in order for the figure to be really fixed. Since we often deal with calibrated cameras, let's fix f, the focal length. Rearranging equation [5] we obtain,

$$\boxed{t = \sin^{-1}(f/f')} \qquad [31]$$

$$\boxed{h = h'\sin t}$$

Now we have fixed figure 1, let's extract the exterior orientation (EO) $\omega, \phi, K, X_L, Y_L, Z_L$. $X_{ref}, Y_{ref}, Z_{ref}$ are the coordinates of origin of u, v system in the reference coordinate system. Refering to figures 1, 2, 3 and the sketch,



$$\boxed{Z_L = h + Z_{ref}} \qquad [32]$$

from figure 2, the segment $\overline{NI}$ is obtained

$$\overline{NI} = h \cdot \tan(t/2) \qquad [33]$$

recall equations [1] & [2], in matrix form,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\alpha' & \sin\alpha' \\ -\sin\alpha' & \cos\alpha' \end{bmatrix}}_{M_{\alpha'}} \begin{bmatrix} u' \\ v' \end{bmatrix} + \begin{bmatrix} d' \\ e' \end{bmatrix} \qquad [5] \; \& \; [34]$$

$$\begin{bmatrix} u \\ v \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}}_{M_{\alpha}} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} d \\ e \end{bmatrix} \qquad [6] \; \& \; [35]$$

now transform N in the xy system into the u,v system,

$$N_{uv} = M_\alpha \begin{bmatrix} -\overline{NI} \\ 0 \end{bmatrix} + \begin{bmatrix} d \\ e \end{bmatrix} \qquad [36]$$

now we get planimetric coordinates of exposure station

$$\begin{bmatrix} X_L \\ Y_L \end{bmatrix} = \begin{bmatrix} X_{ref} \\ Y_{ref} \end{bmatrix} + N_{uv} \qquad [37]$$

Ok, we have exposure station, now get rotation matrix. Recognize that the necessary rotations are in the sequence $M: \begin{pmatrix} u \\ v \end{pmatrix} \underset{①}{\rightarrow} \begin{pmatrix} x \\ y \end{pmatrix} \underset{②}{\rightarrow} \begin{pmatrix} x' \\ y' \end{pmatrix} \underset{③}{\rightarrow} \begin{pmatrix} u' \\ v' \end{pmatrix}$

1. $\begin{pmatrix} u \\ v \end{pmatrix} \rightarrow \begin{pmatrix} x \\ y \end{pmatrix}$ $\quad M_\alpha^T$ (expand to 3D)
2. $\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix}$ $\quad M_y(-t)$ (elementary rotation about y axis)
3. $\begin{pmatrix} x' \\ y' \end{pmatrix} \rightarrow \begin{pmatrix} u' \\ v' \end{pmatrix}$ $\quad M_{\alpha'}^T$ (expand to 3D)

$$M = \begin{bmatrix} \cos\alpha' & -\sin\alpha' & 0 \\ \sin\alpha' & \cos\alpha' & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(-t) & 0 & -\sin(-t) \\ 0 & 1 & 0 \\ \sin(-t) & 0 & \cos(-t) \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad [38]$$

finally extract your favorite angle parameters ω, φ, κ or q_i, q_j, q_k, q_s or azimuth, tilt, swing, etc. We will extract ω, φ, κ (careful if $\phi \approx \pm 90°$)

$$M = \begin{bmatrix} \cos\phi \cos K & \text{———} & \text{———} \\ -\cos\phi \sin K & \text{———} & \text{———} \\ \sin\phi & -\sin\omega \cos\phi & \cos\omega \cos\phi \end{bmatrix} \qquad [39]$$

$$\phi = \sin^{-1} M_{31} \qquad [40]$$

$$\omega = \tan^{-1}\left[{}^{-M_{32}}\!/\!_{M_{33}} \right] \qquad [41]$$

$$K = \tan^{-1}\left[{}^{-M_{21}}\!/\!_{M_{11}} \right] \qquad [42]$$

we are done!

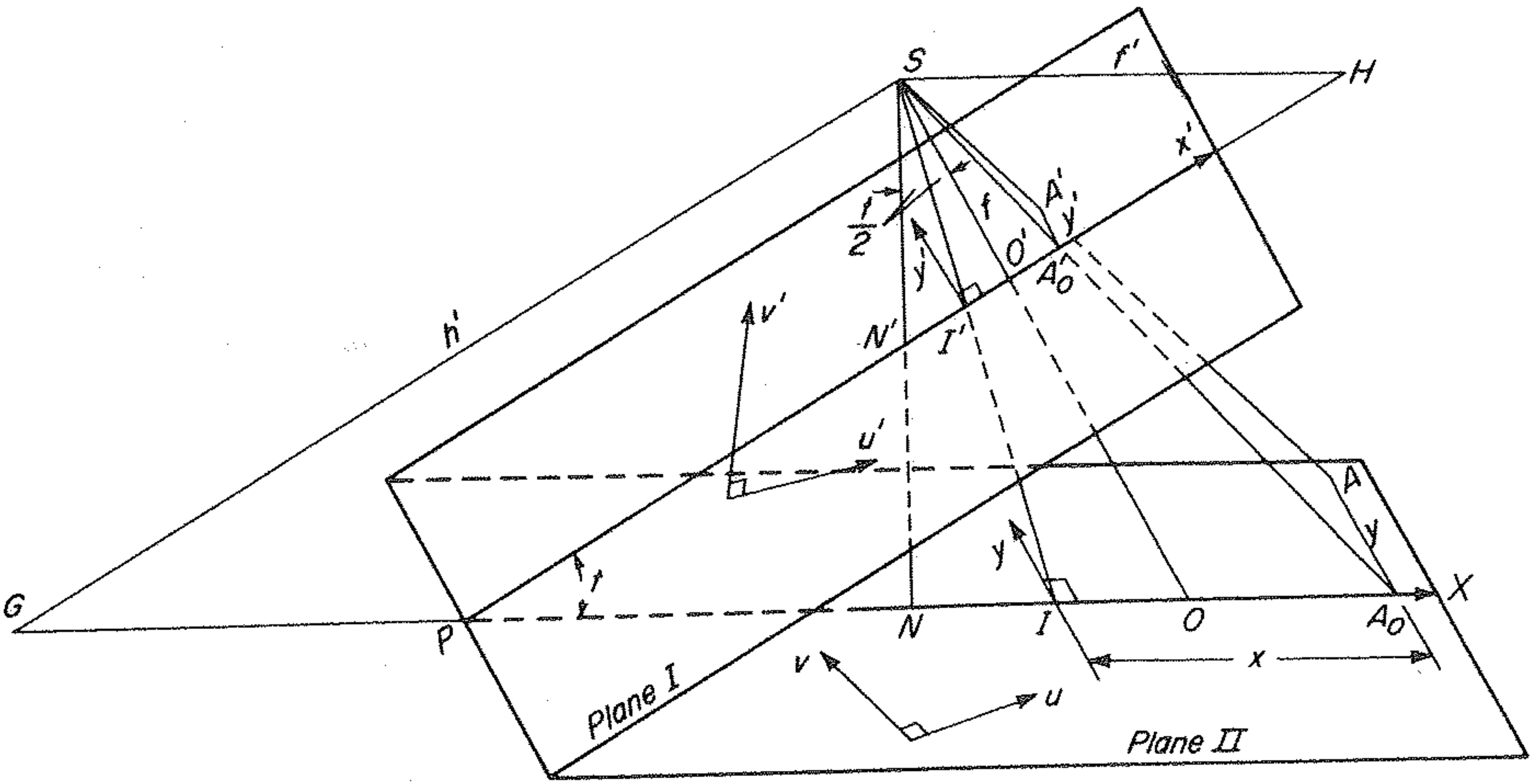


Figure C-1. Projectivity between two planes I and II.

From Photogrammetry by Moffitt & Mikhail

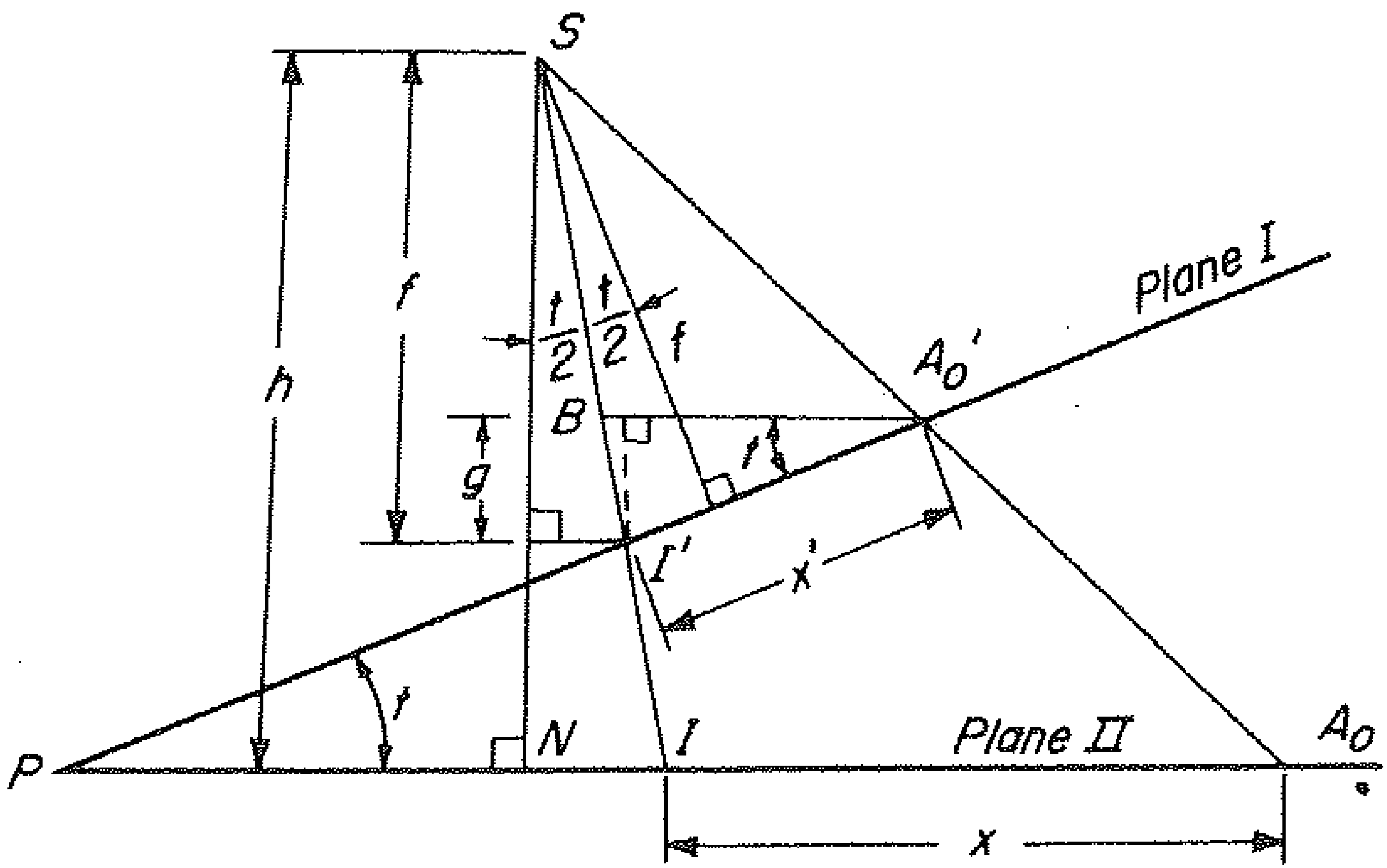


Figure C-2. Principal plane of planes I and II.

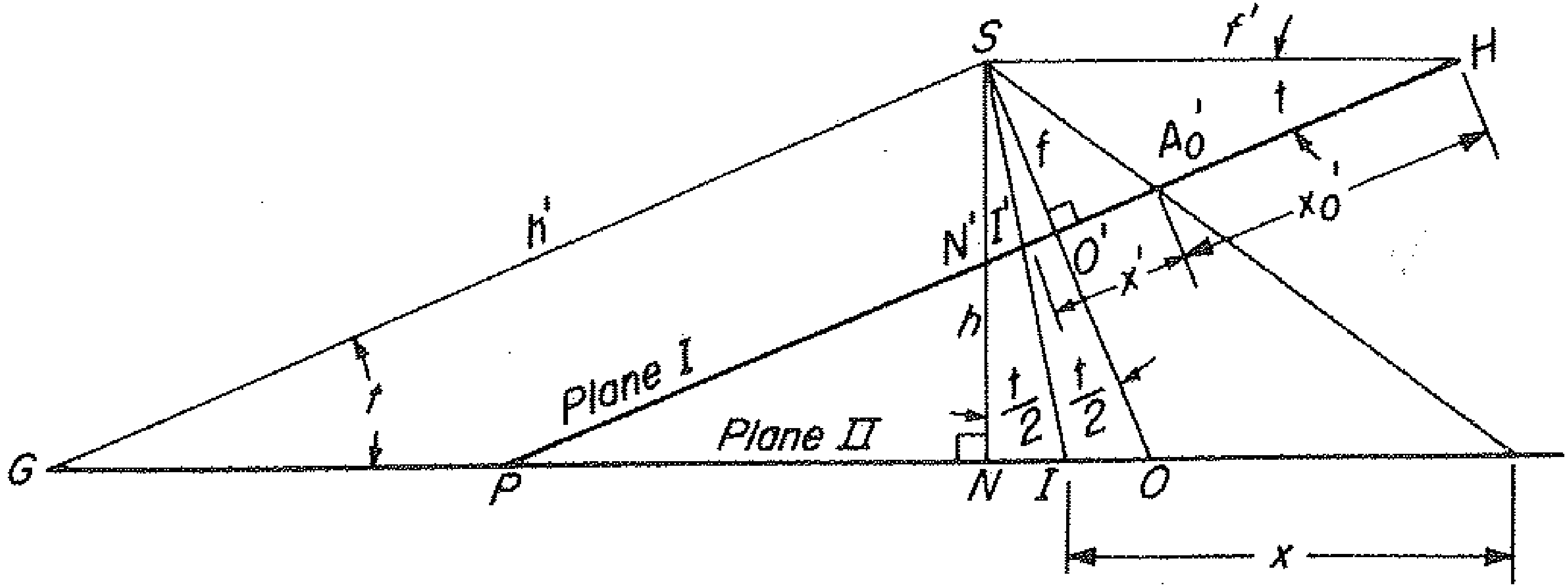


Figure C-3. Principal plane of planes *I* and *II*.