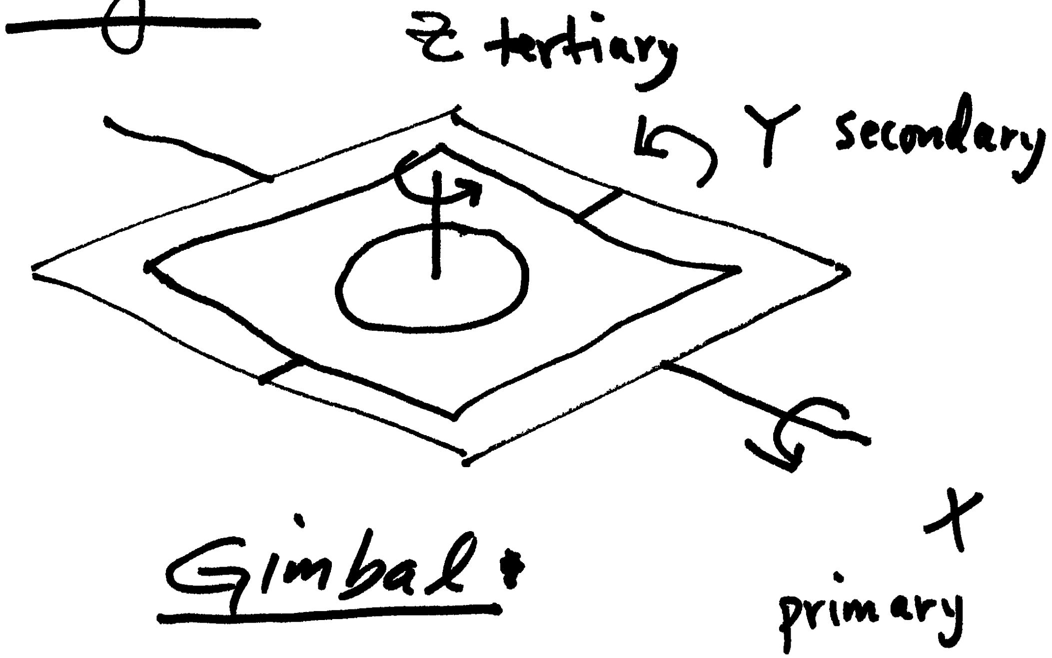
Mechanical analog to Euler Angles

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = M \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{M_\kappa}_{\text{Tertiary}} \, \underbrace{M_\varphi}_{\text{Secondary}} \, \underbrace{M_\omega}_{\text{Primary}} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$



Gimbal

Small angle approximations

θ, δ small angles

- $\cos\theta \approx 1$
- $\sin\theta \approx \theta$ (Radians)
- $\sin\theta \cdot \sin\delta \approx 0$

Approximating rotations

necessary for nonlinear least squares

other applications

geodesy

- datum transformations
- ECF $\leftrightarrow$ ECI

  polar motion rotation

$$M = \begin{bmatrix} c\phi c\kappa & c\omega s\kappa + s\omega s\phi c\kappa & s\omega s\kappa - c\omega s\phi c\kappa \\ -c\phi s\kappa & c\omega c\kappa - s\omega s\phi s\kappa & s\omega c\kappa + c\omega s\phi s\kappa \\ s\phi & -s\omega c\phi & c\omega c\phi \end{bmatrix}$$

small angle approximation, angles are in Radians

$$\begin{bmatrix} 1 & \kappa & -\phi \\ -\kappa & 1 & \omega \\ \phi & -\omega & 1 \end{bmatrix}$$

small angle rotation matrix

same irrespective of order

actual camera orientation M_c

$$M_c \, M_s$$

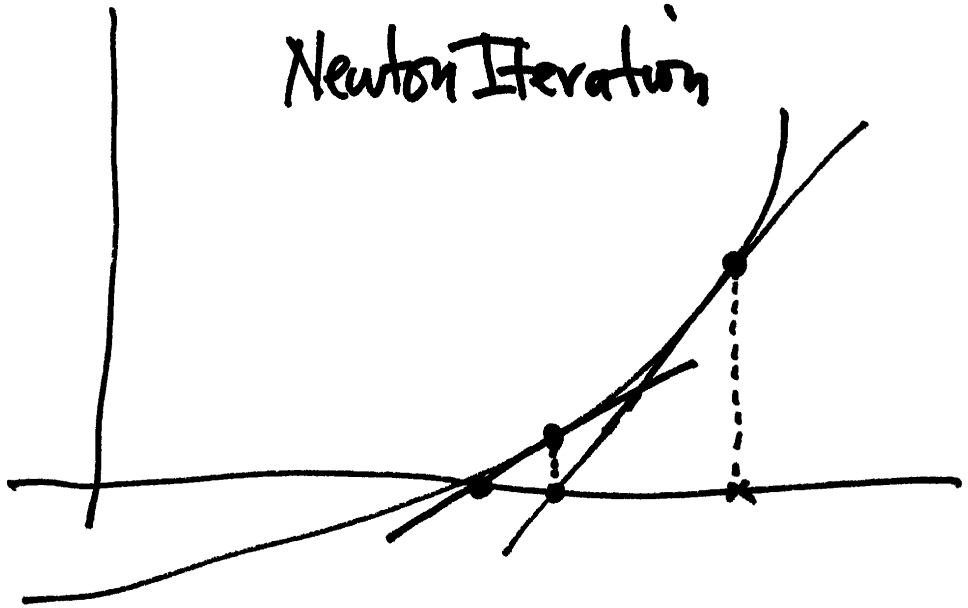
constant

small angle refinement parameters

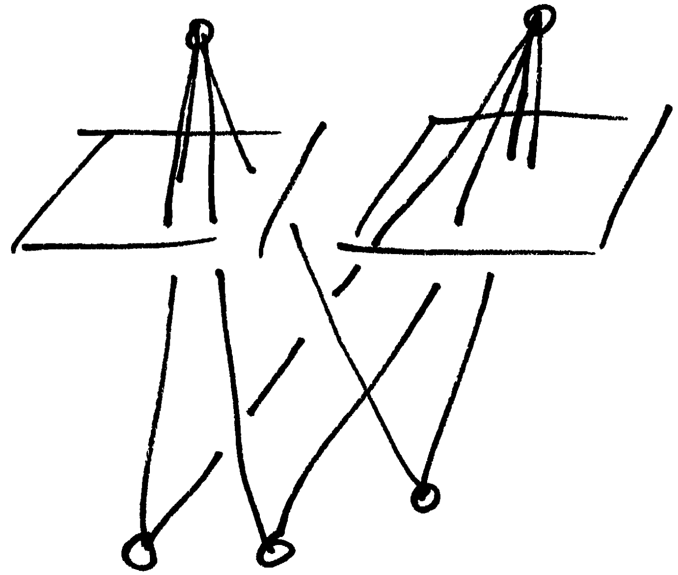
photogrammetric processes

~ always Non-Linear

Newton Iteration



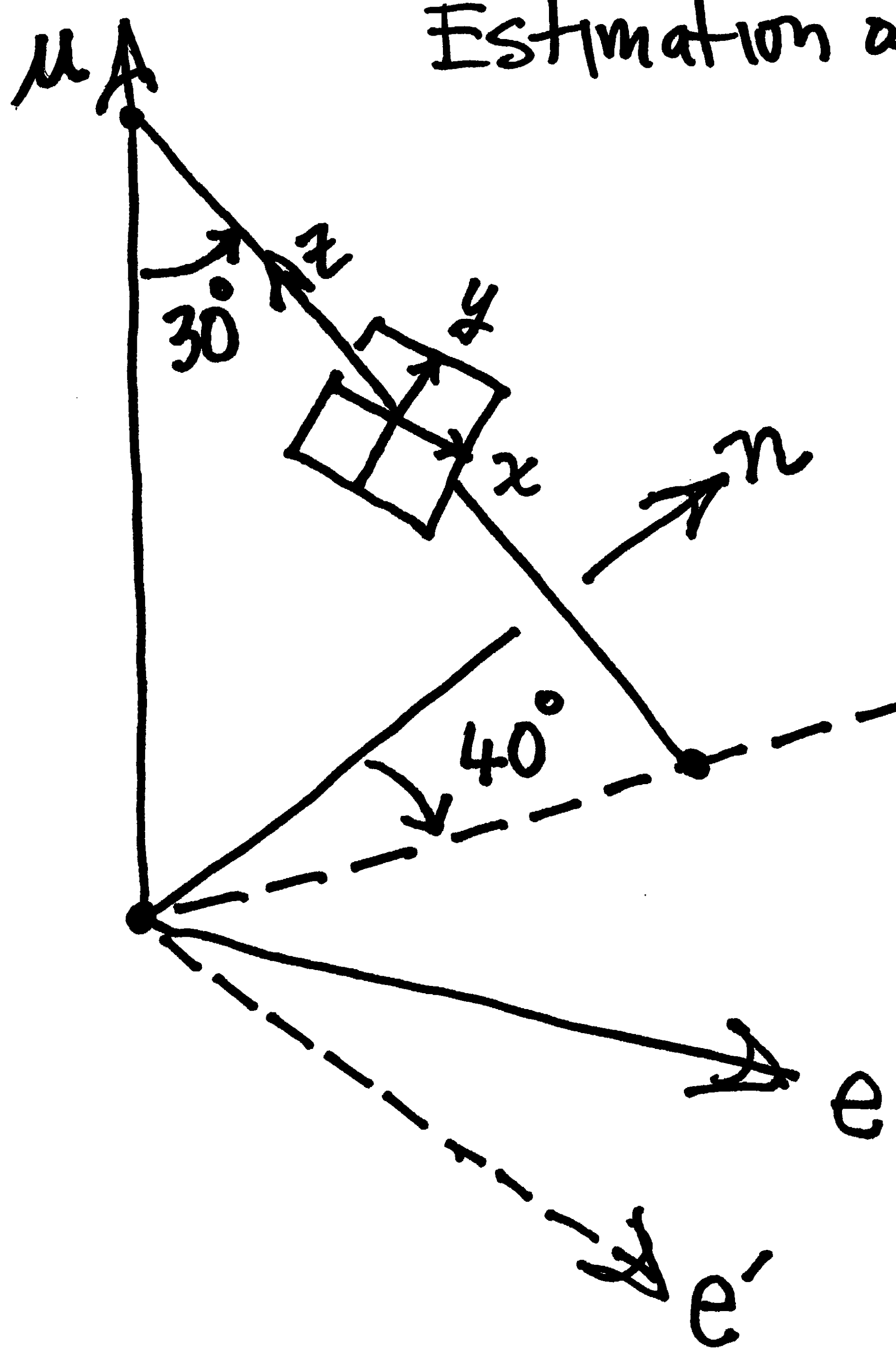
solve for root of non-linear function



$\begin{pmatrix} X_L \\ Y_L \\ Z_L \end{pmatrix}_{1,2}$ $\begin{pmatrix} \omega \\ \varphi \\ \kappa \end{pmatrix}_{1,2}$

$(XYZ)_{1\to3}$

Estimation of M for Oblique Imagery



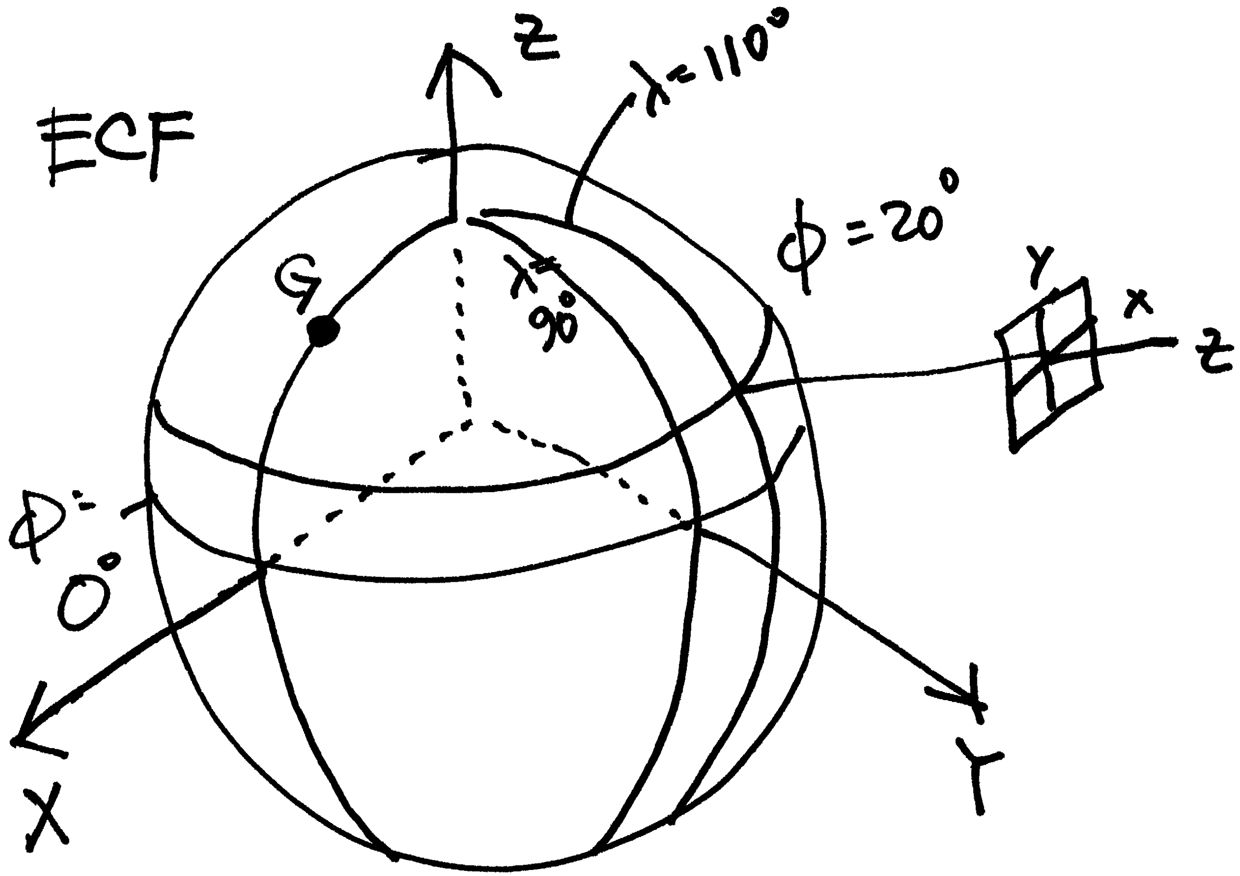
$$M = M_x(+30°) \cdot M_z(-40°)$$

tertiary secondary primary

$$M = M_z M_x M_z$$

common sequence for oblique imagery

ECF



$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = M \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}_{ECF}$$

$$M = M_x(90^\circ - \phi)\, M_z(\lambda + 90^\circ)$$

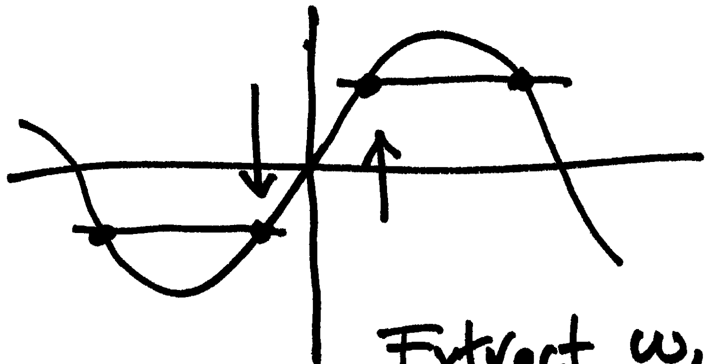
$$\underline{\underline{M}} = M_x(70^\circ)\, M_z(200^\circ)$$

may need to extract

$\omega, \phi, \kappa \qquad M = M_\kappa M_\varphi M_\omega$

ECF: earth centered fixed

$$M = \begin{bmatrix} \cos\phi \cos\kappa & \text{---} & \text{---} \\ -\cos\phi \sin\kappa & \text{---} & \text{---} \\ \sin\phi & -\sin\omega \cos\phi & \cos\omega \cos\phi \end{bmatrix}$$



select one of smaller magnitude

$$\phi = \sin^{-1}(m_{31})$$

Extract ω, ϕ, κ from M

$$\omega = \text{atan}\left(\frac{-m_{32}}{m_{33}}\right) \quad (\tan^{-1})$$

should use $\text{atan2}(-m_{32}, m_{33})$

$$\kappa = \tan^{-1}\left(\frac{-m_{21}}{m_{11}}\right), \text{ use atan2}$$

Start with symbolic M, $\sin(90)=1$, $\cos(90)=0$

$\boxed{\phi = +90^\circ}$ — critical/singular configuration

$$\begin{bmatrix} 0 & \cos\omega\sin\kappa + \sin\omega\cos\kappa & \sin\omega\sin\kappa - \cos\omega\cos\kappa \\ 0 & \cos\omega\cos\kappa - \sin\omega\sin\kappa & \sin\omega\cos\kappa + \cos\omega\sin\kappa \\ 1 & 0 & 0 \end{bmatrix}$$

$\cos(a+b) = \cos a\cos b - \sin a\sin b$, $\sin(a+b) = \sin a\cos b + \cos a\sin b$

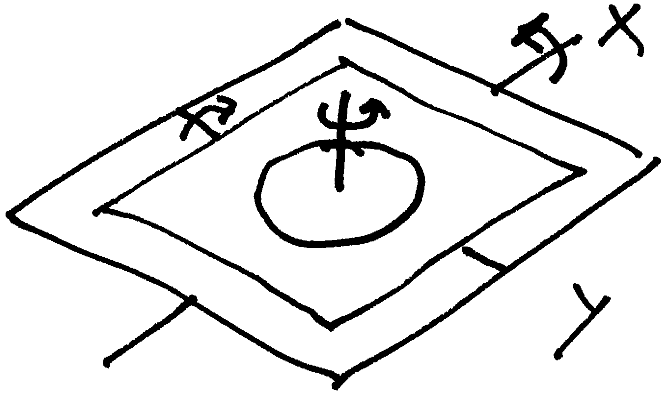
trig identities

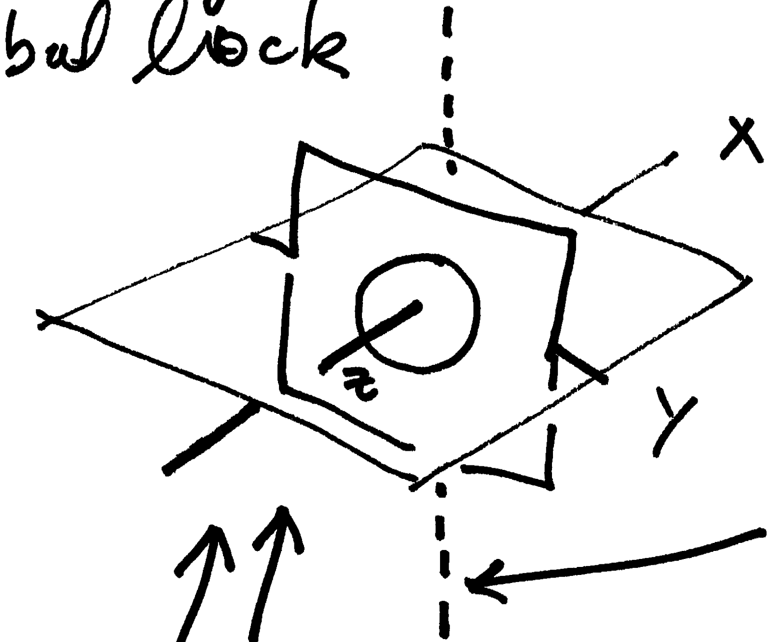
$$\begin{bmatrix} 0 & \sin(\omega+\kappa) & -\cos(\omega+\kappa) \\ 0 & \cos(\omega+\kappa) & \sin(\omega+\kappa) \\ 1 & 0 & 0 \end{bmatrix}$$

$\theta = \omega + \kappa$, cannot solve uniquely for ω & κ

$\phi = +90, -90$ critical configuration
singularity
gimbal lock

<u>INS</u>
stable platform
in critical config
lose D.O.F.





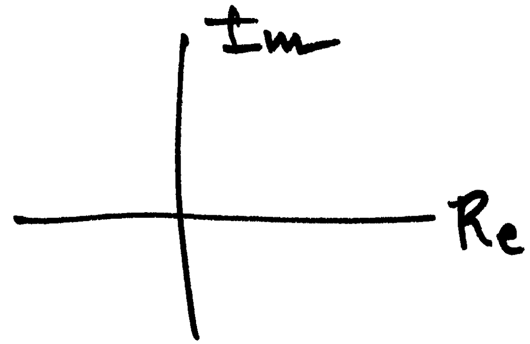
if you need to rotate about ⋮ you cannot do it.

Gimbal: mechanical analogue of euler angles.

X & Z lined up so you lose a rotational degree of freedom

Quaternions

complex $x + iy$



Hamilton 1800's

$$q = q_s + q_i i + q_j j + q_k k$$

$$i^2 = j^2 = k^2 = -1$$

$$ij = k \,,\quad ji = -k$$

$$jk = i \,,\quad kj = -i$$

$$ki = j \,,\quad ik = -j$$

4 dimensional vector

4D "complex numbers"

unit quaternion

$$q_s^2 + q_i^2 + q_j^2 + q_k^2 = 1$$

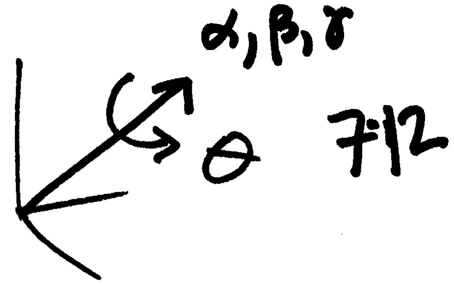
unit quaternion used to build rotation matrix

$$\begin{bmatrix} q_s^2+q_i^2-q_j^2-q_k^2 & 2(q_jq_i-q_sq_k) & 2(q_iq_k+q_sq_j) \\ 2(q_jq_i+q_sq_k) & q_s^2-q_i^2+q_j^2-q_k^2 & 2(q_jq_k-q_sq_i) \\ 2(q_iq_k-q_sq_j) & 2(q_jq_k+q_sq_i) & q_s^2-q_i^2-q_j^2+q_k^2 \end{bmatrix} = M$$

$$\left.\begin{array}{l} a, b, c, d \\ q_i, q_j, q_k, q_s \end{array}\right\}$$

benefit: no singularities

disadvantage: no easy interpretation of q's matrix → axis/angle parameters



$$\theta = \cos^{-1}\left(\frac{\mathrm{tr}\, M - 1}{2}\right)$$

$$\mathrm{tr}(M) = m_{11} + m_{22} + m_{33}$$

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \frac{1}{2\sin\theta}\begin{pmatrix} m_{32} - m_{23} \\ m_{13} - m_{31} \\ m_{21} - m_{12} \end{pmatrix}$$

Conversion from matrix to $\theta, \alpha, \beta, \gamma$

$$\boxed{M \to \theta, \alpha, \beta, \gamma}$$

$\boxed{\alpha\beta\gamma\theta \Rightarrow q}$ conversion

$$q_s = \cos \theta/2$$

$$\begin{pmatrix} q_i \\ q_j \\ q_k \end{pmatrix} = \sin \theta/2 \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

$\boxed{q \Rightarrow \alpha\beta\gamma\theta}$ conversion

$$\cos \theta = q_s^2 - \left(q_i^2 + q_j^2 + q_k^2 \right)$$

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \frac{1}{\left(q_i^2 + q_j^2 + q_k^2 \right)^{1/2}} \begin{pmatrix} q_i \\ q_j \\ q_k \end{pmatrix}$$