4-parameter transf.

$$x = aX + bY + c$$
$$y = -bX + aY + d$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix}$$

apply transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} X & Y & 1 & 0 \\ Y & -X & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

used for estimation

for 2 points:

$$\begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} X_1 & Y_1 & 1 & 0 \\ Y_1 & -X_1 & 0 & 1 \\ X_2 & Y_2 & 1 & 0 \\ Y_2 & -X_2 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$$x = A\,p$$

$$p = A^{-1}x$$

$\lambda, \theta, T_x, T_y$: a, b, c, d

$$a = \lambda\cos\theta$$

$$b = \lambda\sin\theta$$

$$a^2+b^2 = \lambda^2\cos^2\theta + \lambda^2\sin^2\theta$$

$$= \lambda^2(\underbrace{\sin^2\theta + \cos^2\theta}_{1}) = \lambda^2$$

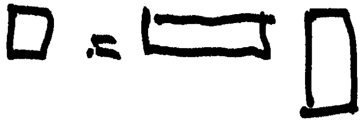
$$\lambda = \sqrt{a^2+b^2} \qquad T_x = c$$

$$\theta = \tan^{-1}(b/a) \qquad T_y = d$$

convert from linear parameters a, b, c, d back to physically meaningful ones $\lambda, \theta, T_x, T_y$

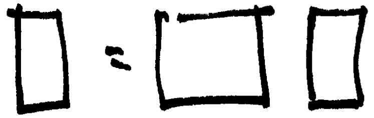
coefficient matrix

m: # rows
n: # cols



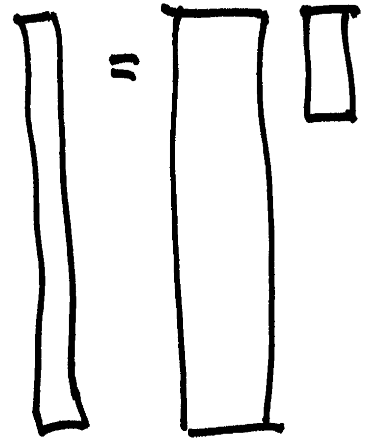
$m < n$

underdetermined
need more info



$m = n$

unique solution



$m > n$

overdetermined
LS

systems of linear equations

3D similarity Transformation

$$x = \lambda M X + T$$
$$(3,1) \quad (3,3) \quad (3,1) \quad (3,1)$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda \underbrace{M}_{\Uparrow} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + \begin{pmatrix} T_x \\ T_y \\ T_z \end{pmatrix}$$

λ, ω, φ, K, T_x, T_y, T_z

q_i, q_j, q_k

α, β, θ

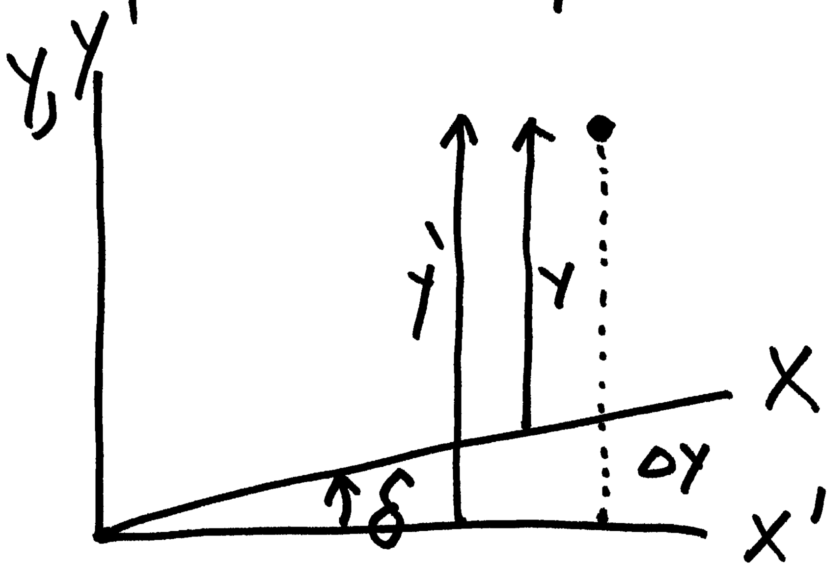
7-parameter transformation

conformal, rigid body,

shape preserved

affine 2D transformation, 6 parameter transf.

Scale x, scale y, 2 shifts, rotation, non-orthogonality



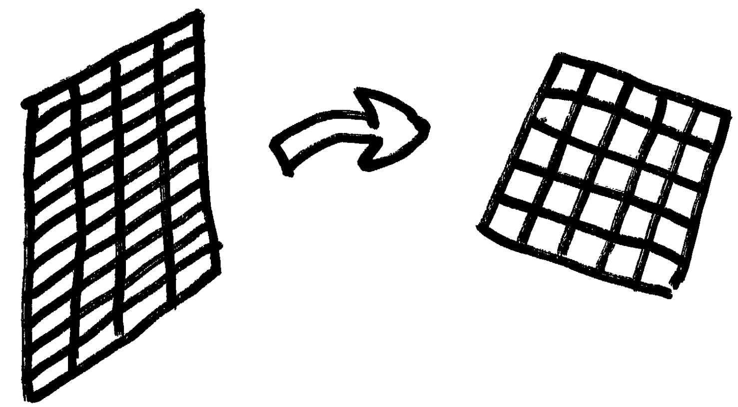
assume δ is small

$$\frac{x'}{X} = \cos\delta \approx 1$$

$$\frac{\Delta Y}{X} = \sin\delta \approx \delta \text{ (radians)}$$

$$\left.\begin{aligned} x' &= X \\ y' &= y + \delta x \end{aligned}\right\} \quad \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ \delta & 1 \end{bmatrix} \begin{pmatrix} X \\ y \end{pmatrix}$$

Scale



$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \underset{\uparrow \\ \text{rot}}{\begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix}} \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad \underset{\uparrow \\ \text{trans}}{}$$

$$\begin{pmatrix} x''' \\ y''' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix} \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \end{pmatrix}$$

6 par. transf. with physical parameters $\theta, \delta, s_x, s_y, t_x, t_y$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} S_x \cos\theta + \delta \sin\theta & S_y \sin\theta \\ S_x(-\sin\theta + \delta\cos\theta) & S_y \cos\theta \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \end{pmatrix}$$

$a_1 \qquad b_1 \qquad a_2 \qquad b_2 \qquad a_0 \qquad b_0$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \left.\begin{matrix} a_0 + a_1 X + a_2 Y \\ b_0 + b_2 X + b_2 Y \end{matrix}\right\} \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} + \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

application

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & X & Y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & X & Y \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ b_0 \\ b_1 \\ b_2 \end{pmatrix}$$

estimation

$$S_y = \sqrt{a_2^2 + b_2^2}$$

$$\theta = \tan^{-1}(a_2/b_2)$$

$$S_x = \frac{a_1 b_2 - b_1 a_2}{S_y}$$

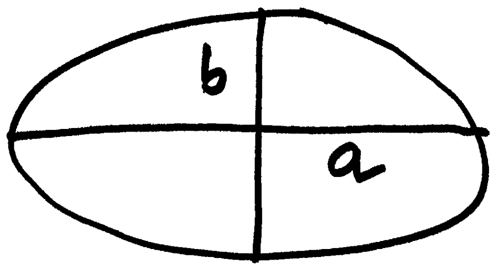
$$\delta = \frac{a_1 - S_x \cos\theta}{S_x \sin\theta}$$

$$t_x = a_0$$

$$t_y = b_0$$

can also use a 2D 8-parameter transformation

WGS 84 reference ellipsoid



a, b

a, 1/f

$$f = \frac{a-b}{a} \quad \text{flattening}$$

$$e^2 = \frac{a^2 - b^2}{a^2}$$

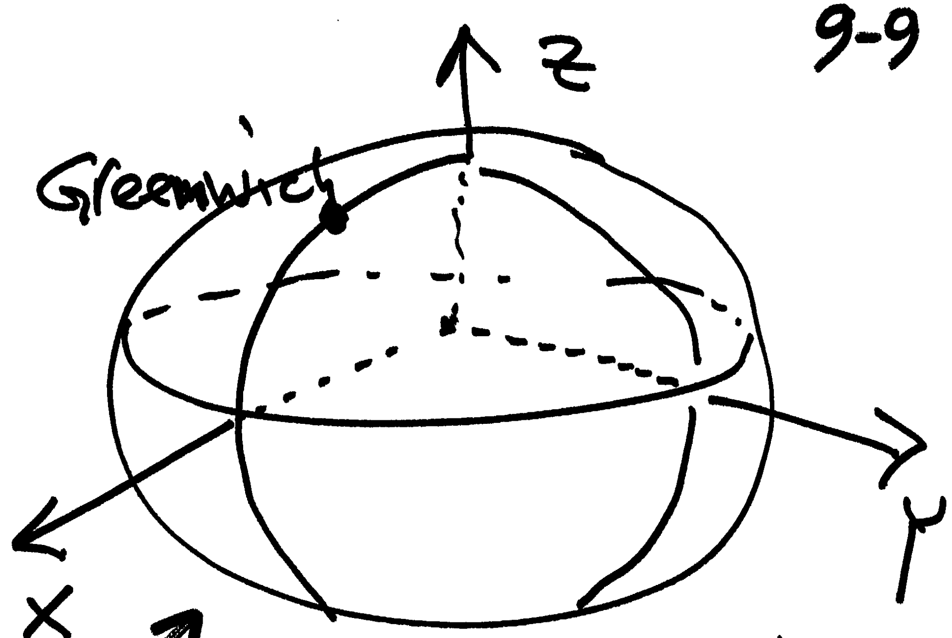
$$e^2 = 2f - f^2$$

$$b^2 = a^2(1-e^2)$$

$$a = 6378137.0 \text{ m}$$

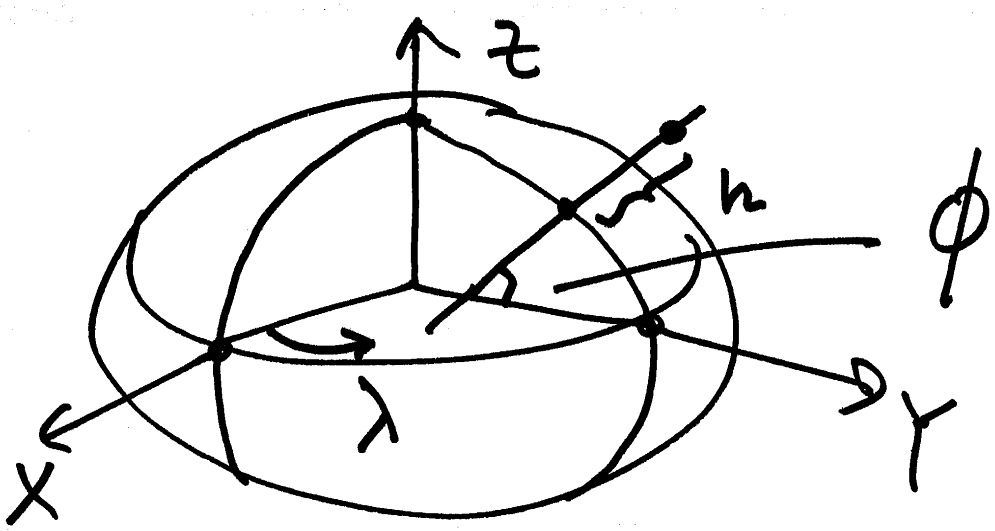
$$1/f = 298.257223563$$

$$a - b \approx 21 \text{ km}$$



ECF earth centered fixed coordinate system

ECI
earth centered inertial : X fixed to stars (vernal equinox)



$$\begin{pmatrix}\phi\\ \lambda\\ h\end{pmatrix} \longleftrightarrow \begin{pmatrix}X\\ Y\\ Z\end{pmatrix}$$

$$\begin{pmatrix}X\\ Y\\ Z\end{pmatrix} = \begin{bmatrix}(N+h)\cos\phi\cos\lambda\\ (N+h)\cos\phi\sin\lambda\\ ((1-e^2)N+h)\sin\phi\end{bmatrix}$$

closed form

$$N = \frac{a}{(1-e^2\sin^2\phi)^2}$$

radius of curvature in prime vertical

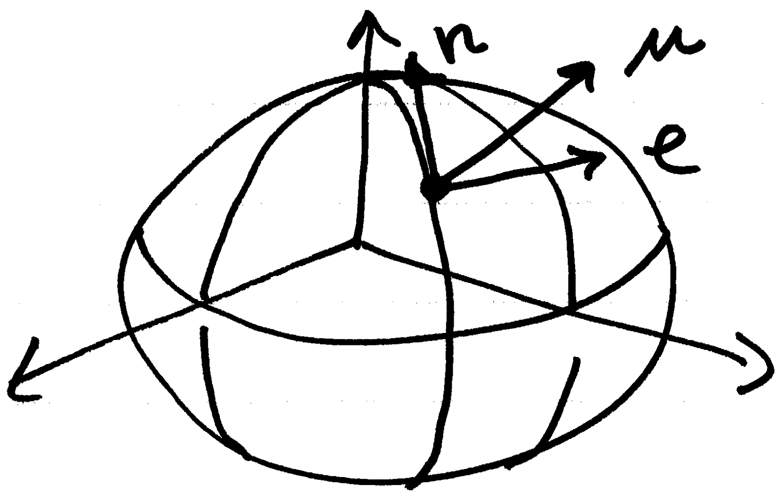
start with ϕ° $\quad$ $\begin{matrix}X\\Y\\Z\end{matrix} \rightarrow \phi, \lambda, h$, iterative

$\lambda = \tan^{-1}(Y/X)$ $\quad$ atan2, resolve quadrant

$$\phi = \tan^{-1}\left[\frac{Z}{(X^2+Y^2)^{1/2}}\left(1-e^2\left(\frac{N}{N+h}\right)\right)^{-1}\right]$$

$$h = \frac{(X^2+Y^2)^{1/2}}{\cos\phi} - N$$

iterate until change negligible



local cartesian
topocentric