Least Squares $\nearrow$ Estimation, $\searrow$ Error Propagation

$$y = Ax, \quad \Sigma_{xx}$$

$$\Sigma_{yy} = A \Sigma_{xx} A^T$$

$$Q_{\hat{\delta}\hat{\delta}} = N^{-1}$$ Individ Obs.

$$\Sigma = \sigma_0^2 Q, \quad N = B^T W B$$

$$Q_{\hat{\ell}\hat{\ell}} = B N^{-1} B^T$$

$$\Sigma_{ee} \to W_{ee}$$

$$Q_{vv} = Q - B N^{-1} B^T$$

only valid if results of adjustment are consistent with expectations

$\Rightarrow$ no blunders + no systematic errors confirm consistency : statistical test Global Test on Reference Variance

test statistic $= G = \dfrac{v^T W v}{\sigma_0^2}$ $W = \sigma_0^2 \Sigma^{-1}$

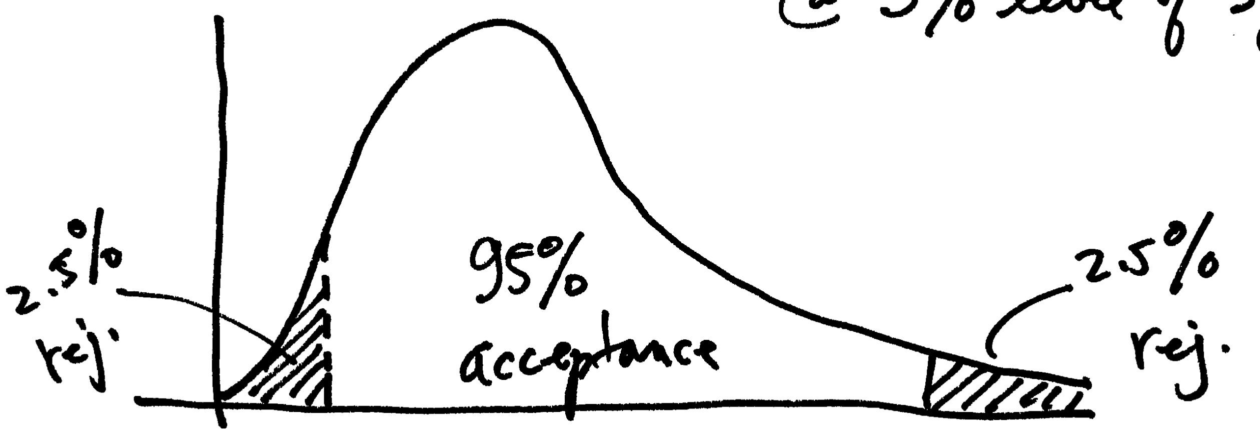
$G = \dfrac{v^T \cancel{\sigma_0^2} \Sigma^{-1} v}{\cancel{\sigma_0^2}} = \dfrac{v^T \Sigma^{-1} v}{1}$ $\left\{ \text{G does not depend on choice of } \sigma_0^2 \right\}$

$H_0 : \sigma^2 = \sigma_0^2$

$H_1 : \sigma^2 \neq \sigma_0^2$

under null hyp. $G \sim \chi^2_r$ r: redundancy

@ 5% level of Sign.



Taxonomy LS techniques

- observations only — not used much
- ✓ indirect observations — most common, n cond. eqn's, $u = n_0$
- ✓ general LS (mixed model) — # parameters $< n_0$, $c = r + u$
- constrained LS — enforce relations between parameters
- unified LS — prior knowledge value + σ parameter
- sequential LS
- Kalman Filter — seq LS + dynamic model transition matrix

<u>6 par transf.</u> <u>5 pts</u> xy obs, XY constant

$$x = a_0 + a_1 X + a_2 Y$$
$$y = b_0 + b_1 X + b_2 Y$$

$\rightarrow$ $V + B\Delta = f$

$n = 10$ ind obs
$n_0 = 6$ $u = n_0 = 6$
$r = 4$ $c = n = 10$

$$x_i + v_{x_i} - a_0 - a_1 X_i - a_2 Y_i = 0$$
$$y_i + v_{y_i} - b_0 - b_1 X_i - b_2 Y_i = 0$$

$$v_{x_i} - a_0 - a_1 X_i - a_2 Y_i = -x_i$$
$$v_{y_i} - b_0 - b_1 X_i - b_2 Y_i = -y_i$$

constant obs

$$\underset{n,1}{\begin{bmatrix} V_{x_1} \\ V_{y_1} \\ V_{x_2} \\ V_{y_2} \\ \vdots \\ V_{x_5} \\ V_{y_5} \end{bmatrix}} + \underset{n,u}{\begin{bmatrix} -1 & -X_1 & -Y_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -X_1 & -Y_1 \\ -1 & -X_2 & -Y_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -X_1 & -Y_1 \\ & & & \vdots & & \\ -1 & -X_5 & -Y_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -X_5 & -Y_5 \end{bmatrix}} \underset{u,1}{\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ b_0 \\ b_1 \\ b_2 \end{bmatrix}} = \underset{n,1}{\begin{bmatrix} -x_1 \\ -y_1 \\ -x_2 \\ -y_2 \\ \vdots \\ -x_5 \\ -y_5 \end{bmatrix}}$$

$$V + B\Delta = f$$

$$\Delta = (B^T W B)^{-1} B^T W f = N^{-1} t$$

$$v = f - B\Delta$$

$$\hat{l} = l + v$$

$$G = \frac{v^T W v}{\sigma_0^2}$$ test

$Q_{\Delta\Delta}$

$\Sigma_{\Delta\Delta}$

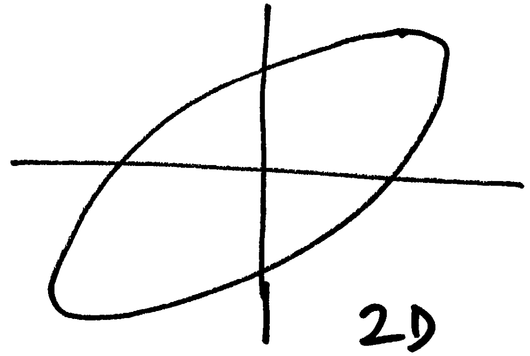
confidence intervals

confidence region

confidence interval (1D)

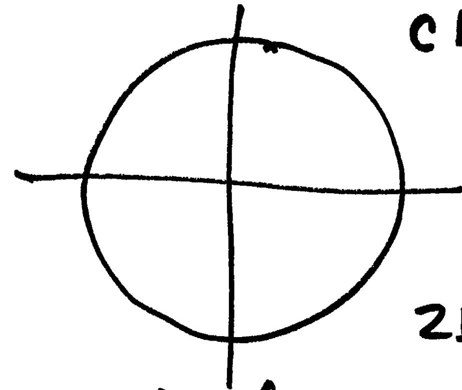
CE

CE90



2D

elliptical error



2D

circular error

<u>8-par transf.</u> $x = \dfrac{a_0 + a_1 X + a_2 Y}{1 + c_1 X + c_2 Y}$, $y = \dfrac{b_0 + b_1 X + b_2 Y}{1 + c_1 X + c_2 Y}$

6 points $n = 12$ ind. obs $u = 8$ (n_0)

$n_0 = 8$ $c = 12$ (n)

$r = 4$

$$v_{x_i} - a_0 - a_1 X_i - a_2 Y_i - x_i c_1 X_i - x_i c_2 Y_i = -x_i$$

$$v_{y_i} - b_0 - b_1 X_i - b_2 Y_i - y_i c_1 X_i - y_i c_2 Y_i = -y_i$$

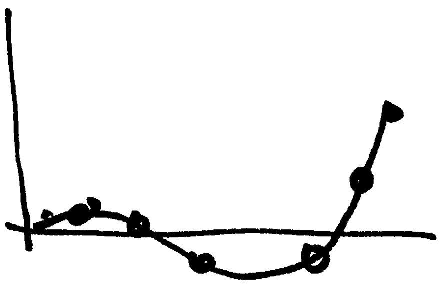
pseudo linear cond. equations.

In this form we can just read by inspection coefficients for $V + B\Delta = f$

$$\begin{bmatrix} V_{x_1} \\ V_{y_1} \\ V_{x_2} \\ V_{y_2} \\ \vdots \\ V_{x_6} \\ V_{y_6} \end{bmatrix} + \begin{bmatrix} -1 & -X_1 & -Y_1 & 0 & 0 & 0 & -x_1X_1 & -x_1Y_1 \\ 0 & 0 & 0 & -1 & -X_1 & -Y_1 & -y_1X_1 & -y_1Y_1 \\ -1 & -X_2 & -Y_2 & 0 & 0 & 0 & -x_2X_2 & -x_2Y_2 \\ 0 & 0 & 0 & -1 & -X_2 & -Y_2 & -y_2X_2 & -y_2Y_2 \\ & & & \vdots & & & & \\ -1 & -X_6 & -Y_6 & 0 & 0 & 0 & -x_6X_6 & -x_6Y_6 \\ 0 & 0 & 0 & -1 & -X_6 & -Y_6 & -y_6X_6 & -y_6Y_6 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ b_0 \\ b_1 \\ b_2 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -x_1 \\ -y_1 \\ -x_2 \\ -y_2 \\ \vdots \\ -x_6 \\ -y_6 \end{bmatrix}$$

$$V + B\Delta = f$$

$\vdots$



fit polynomial to lens distortion curve at 5 discrete points

$$c = \max r$$

$$d = k_1 r^3 + k_2 r^5 + K_3 r^7$$

condition number (CN) $= \dfrac{\lambda_{max}}{\lambda_{min}}$ of N

λ : $Ax = \lambda x$, λ: eigenvalue

if CN $> 10^{15}$ you have solution stability problem

$$v + d = k_1 \frac{r^3}{c^3} + k_2 \frac{r^5}{c^5}, \quad v - k_1 \frac{r^3}{c^3} - k_2 \frac{r^5}{c^5} = -d$$

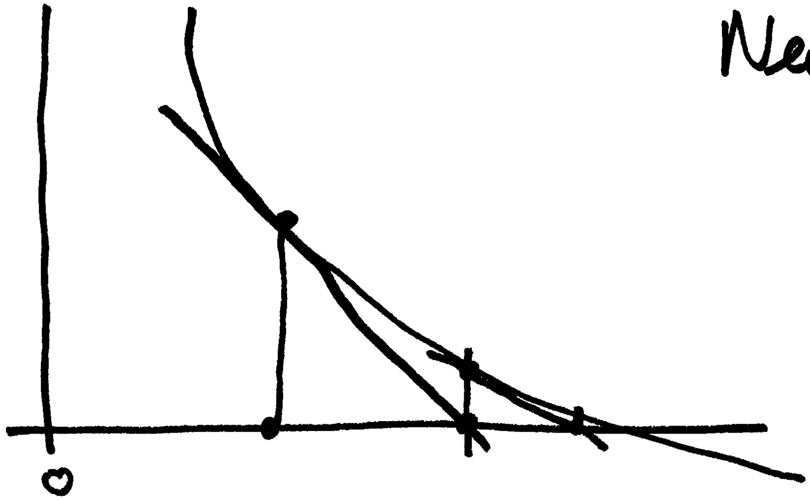
$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_5 \end{bmatrix} + \begin{bmatrix} -\frac{r_1^3}{c^3} & -\frac{r_1^5}{c^5} \\ -\frac{r_2^3}{c^3} & -\frac{r_2^5}{c^5} \\ & \vdots \\ -\frac{r_5^3}{c^3} & -\frac{r_5^5}{c^6} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} -d_1 \\ -d_2 \\ \vdots \\ -d_5 \end{bmatrix}$$

$$V + B\Delta = f$$

you can help to keep the CN within reasonable bounds, sometimes, by scaling, as shown in this example.

Non Linear LS

Newton Iteration Review



1D

$$y = mx + b$$
$$0 = mx + b$$
$$mx = -b$$
$$x = -b/m$$

$$F(x) = F'(x_0)(x - x_0) + F(x_0)$$
$$0 = F'(x_0)(x - x_0) + F(x_0)$$
$$-F(x_0) = F'(x_0)(x - x_0)$$
$$x - x_0 = \frac{-F(x_0)}{F'(x_0)}$$

$$\boxed{x = x_0 + \underbrace{\frac{-F(x_0)}{F'(x_0)}}_{\Delta x}}$$

This is the picture to keep in mind when trying to visualize non linear least squares, NLLS, iterations, since we cannot easily visualize higher dimensions.

$$x_{i+1} = x_i + \underbrace{\frac{-F(x_i)}{F'(x_i)}}_{\Delta x}$$

repeat until Δx small

iteration formula, 1 equation 1 unknown

$$\left.\begin{aligned} F_1(x_1, x_2) &= 0 \\ F_2(x_1, x_2) &= 0 \end{aligned}\right\}$$

let's look at an example with 2 equations in 2 unknowns.

$$0 = F_1(X_1, X_2) = F_1(X_1^0, X_2^0) + \frac{\partial F_1}{\partial X_1}\Delta X_1 + \frac{\partial F_1}{\partial X_2}\Delta X_2 + \cdots$$

$$0 = F_2(X_1, X_2) =$$

finish this next time ....