Longhand Solution of LS regression problem from class 2 Sept 2005

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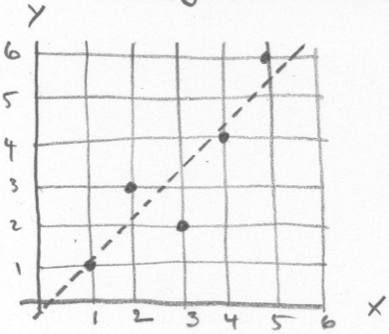

x	y
1	1
2	3
3	2
4	4
5	6

fit line ($y = a_0 + a_1 x$) to data points

y: observation, x: constant

$n = 5$

$n_0 = 2$

$r = 3$

$y_i + v_i = a_0 + a_1 x_i$

a_0, a_1 : unknown parameters

observations are of equal precision and uncorrelated i.e. do not need weights.

$\Phi = \sum v_i^2 \rightarrow \text{minimum}$, $v_1^2 + v_2^2 + v_3^2 + v_4^2 + v_5^2 \rightarrow \text{minimum}$

$v_i = a_0 + a_1 x_i - y_i$

$$\left.\begin{aligned}
v_1 &= a_0 + a_1 \cdot 1 - 1 \\
v_2 &= a_0 + a_1 \cdot 2 - 3 \\
v_3 &= a_0 + a_1 \cdot 3 - 2 \\
v_4 &= a_0 + a_1 \cdot 4 - 4 \\
v_5 &= a_0 + a_1 \cdot 5 - 6
\end{aligned}\right\}$$

$$\Phi = (a_0 + a_1 \cdot 1 - 1)^2 + (a_0 + a_1 \cdot 2 - 3)^2 + (a_0 + a_1 \cdot 3 - 2)^2 + (a_0 + a_1 \cdot 4 - 4)^2 + (a_0 + a_1 \cdot 5 - 6)^2 \rightarrow \text{minimum}$$

by calculus, differentiate and set = zero

$$\frac{\partial \Phi}{\partial a_0} = \cancel{2}(a_0 + a_1 \cdot 1 - 1) + \cancel{2}(a_0 + a_1 \cdot 2 - 3) + \cancel{2}(a_0 + a_1 \cdot 3 - 2) + \cancel{2}(a_0 + a_1 \cdot 4 - 4) + \cancel{2}(a_0 + a_1 \cdot 5 - 6) = 0$$

$$\frac{\partial \Phi}{\partial a_1} = \cancel{2}(a_0 + a_1 \cdot 1 - 1) \cdot 1 + \cancel{2}(a_0 + a_1 \cdot 2 - 3) \cdot 2 + \cancel{2}(a_0 + a_1 \cdot 3 - 2) \cdot 3 + \cancel{2}(a_0 + a_1 \cdot 4 - 4) \cdot 4 + \cancel{2}(a_0 + a_1 \cdot 5 - 6) \cdot 5 = 0$$

combine terms and simplify these 2 equations and we get 2 equations in 2 unknowns (a_0, a_1) which we can solve by gauss elimination, or by matrix inversion,

$$\begin{aligned} 5a_0 + 15a_1 - 16 &= 0 \\ 15a_0 + 55a_1 - 59 &= 0 \end{aligned}, \quad \begin{aligned} 5a_0 + 15a_1 &= 16 \\ 15a_0 + 55a_1 &= 59 \end{aligned}, \quad \text{matrix form: } \begin{bmatrix} 5 & 15 \\ 15 & 55 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 16 \\ 59 \end{bmatrix}$$

These are called the normal equations, note matrix is symmetric

Solve by gauss elimination,

$a_0 + \frac{15}{5} a_1 = 16/5$, $a_0 = 3.2 - 3a_1$

$15(3.2 - 3 \cdot a_1) + 55a_1 = 59$

$48 - 45a_1 + 55a_1 = 59$

$10a_1 = 11$, $a_1 = 1.1$,

$a_0 = 3.2 - 3 \cdot 1.1 = -0.1$

$[a_0, a_1] = [-0.1, 1.1]$

Solve by matrix inversion,

$$\begin{bmatrix} 5 & 15 \\ 15 & 55 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 16 \\ 59 \end{bmatrix} \quad , \quad N x = t \quad , \quad x = N^{-1} t$$

$$N^{-1} = \frac{\text{adjoint}(N)}{\det(N)} = \frac{[\text{cofactor}(N)]^T}{\det(N)}$$

cofactor element $i,j = (-1)^{i+j} \times \det(\text{matrix without row } i, \text{col } j)$

$$N^{-1} = \frac{\begin{bmatrix} 55 & -15 \\ -15 & 5 \end{bmatrix}^T}{(5 \times 55 - 15^2)} = \begin{bmatrix} 1.1 & -0.3 \\ -0.3 & 0.1 \end{bmatrix}$$

$$x = N^{-1} t = \begin{bmatrix} 1.1 & -.3 \\ -.3 & .1 \end{bmatrix} \begin{bmatrix} 16 \\ 59 \end{bmatrix} = \begin{bmatrix} -0.1 \\ 1.1 \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$$ same as prior solution

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Solve by matlab,

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N = [5 15 ; 15 55] ;
t = [16 ; 59] ;
x = inv(N) * t ;
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$$x = \begin{bmatrix} -0.1 \\ 1.1 \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$$

Excel

	A	B
1	1	1
2	2	3
3	3	2
4	4	4
5	5	6

= slope (B1:B5, A1:A5)

= intercept (B1:B5, A1:A5)

	A	B	C	D
7	5	15		16
8	15	55		59
9				
10	1.1	-.3		-.1
11	-.3	.1		1.1

select destination

=minverse (A7:B8) ctrl-shift-enter

select destination

= mmult (A10:B11, D7:D8) ctrl-shift-enter

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compute residuals, plug in $x, y, \hat{a}_0, \hat{a}_1$

$$V_i = \hat{a}_0 + \hat{a}_1 \cdot x_i - y_i$$

$V_1 = 0$

$V_2 = -.9$

$V_3 = 1.2$

$V_4 = .3$

$V_5 = -.6$

compute adjusted observations, $\hat{y}_i = y_i + V_i$

$\hat{y}_1 = 1.0$

$\hat{y}_2 = 2.1$

$\hat{y}_3 = 3.2$

$\hat{y}_4 = 4.3$

$\hat{y}_5 = 5.4$

This LS solution is done by the method of indirect observations, i.e. each observation is expressed as function of no parameters

$$Y = a_0 + a_1 x$$