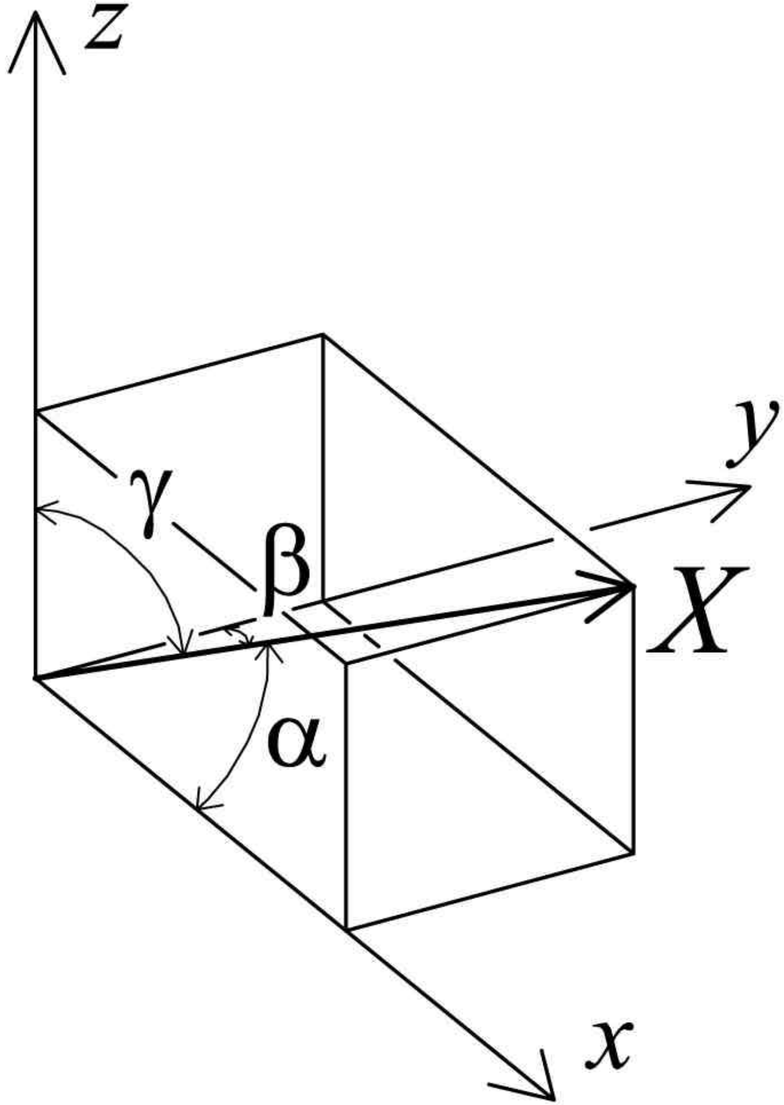
Direction Cosines



$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{M} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

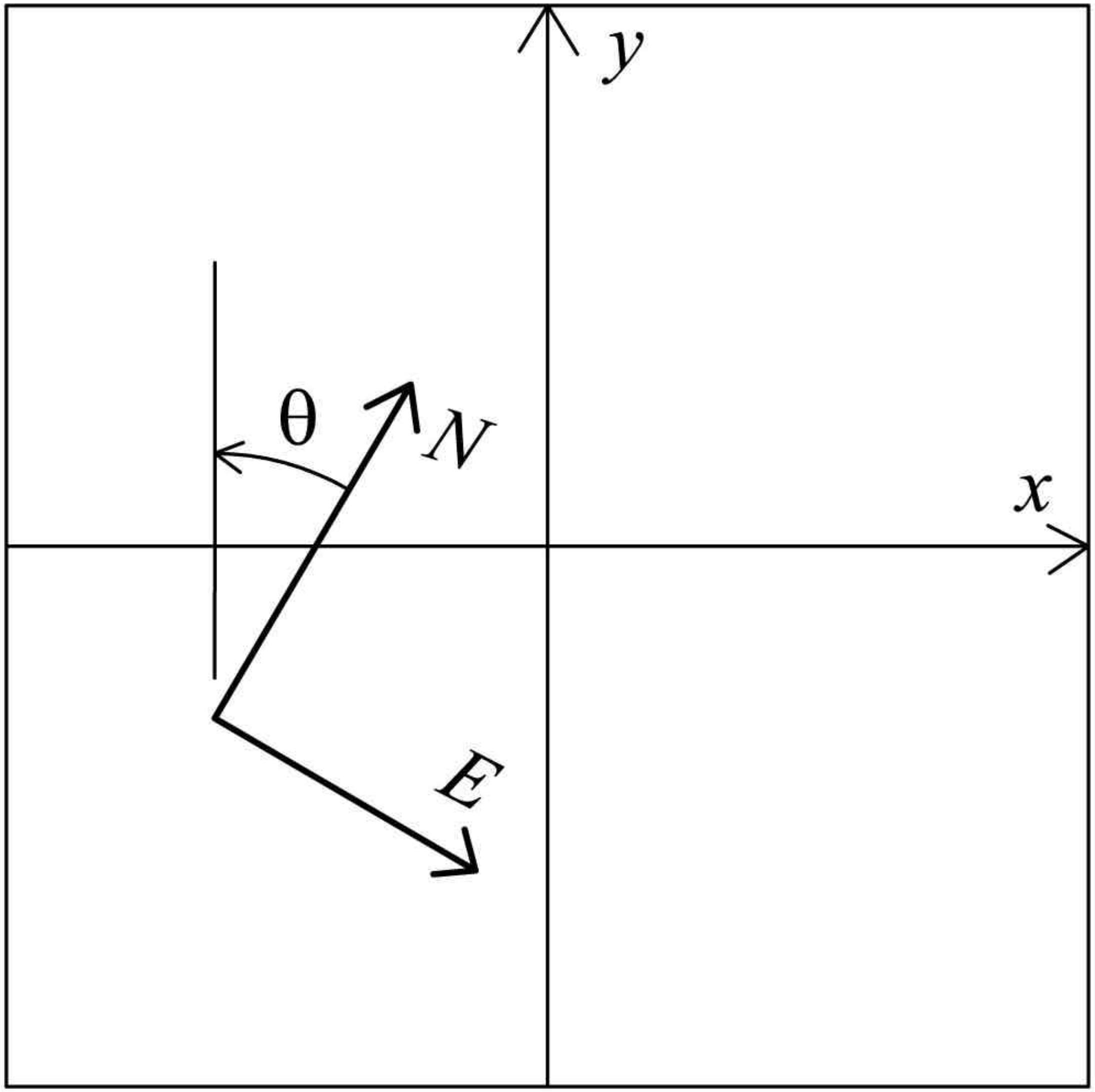
consider the lower case image of the unit vector [1,0,0] on the right

$$\begin{bmatrix} m_{11} \\ m_{21} \\ m_{31} \end{bmatrix} = \mathbf{M} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos a \\ \cos b \\ \cos g \end{bmatrix} = \begin{bmatrix} \cos xX \\ \cos yX \\ \cos zX \end{bmatrix}$$

therefore the full matrix M expressed via direction cosines would be

$$\mathbf{M} = \begin{bmatrix} \cos xX & \cos xY & \cos xZ \\ \cos yX & \cos yY & \cos yZ \\ \cos zX & \cos zY & \cos zZ \end{bmatrix}$$

Example 1, approximate the matrix



$$q_z = k \approx +30^\circ$$

$$\mathbf{M} = \mathbf{M}_k(30) =$$

$$\begin{bmatrix} \cos k & \sin k & 0 \\ -\sin k & \cos k & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 0.866 & 0.5 & 0 \\ -0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{M} \begin{bmatrix} E \\ N \\ h \end{bmatrix}$$

Example 2, approximate the matrix

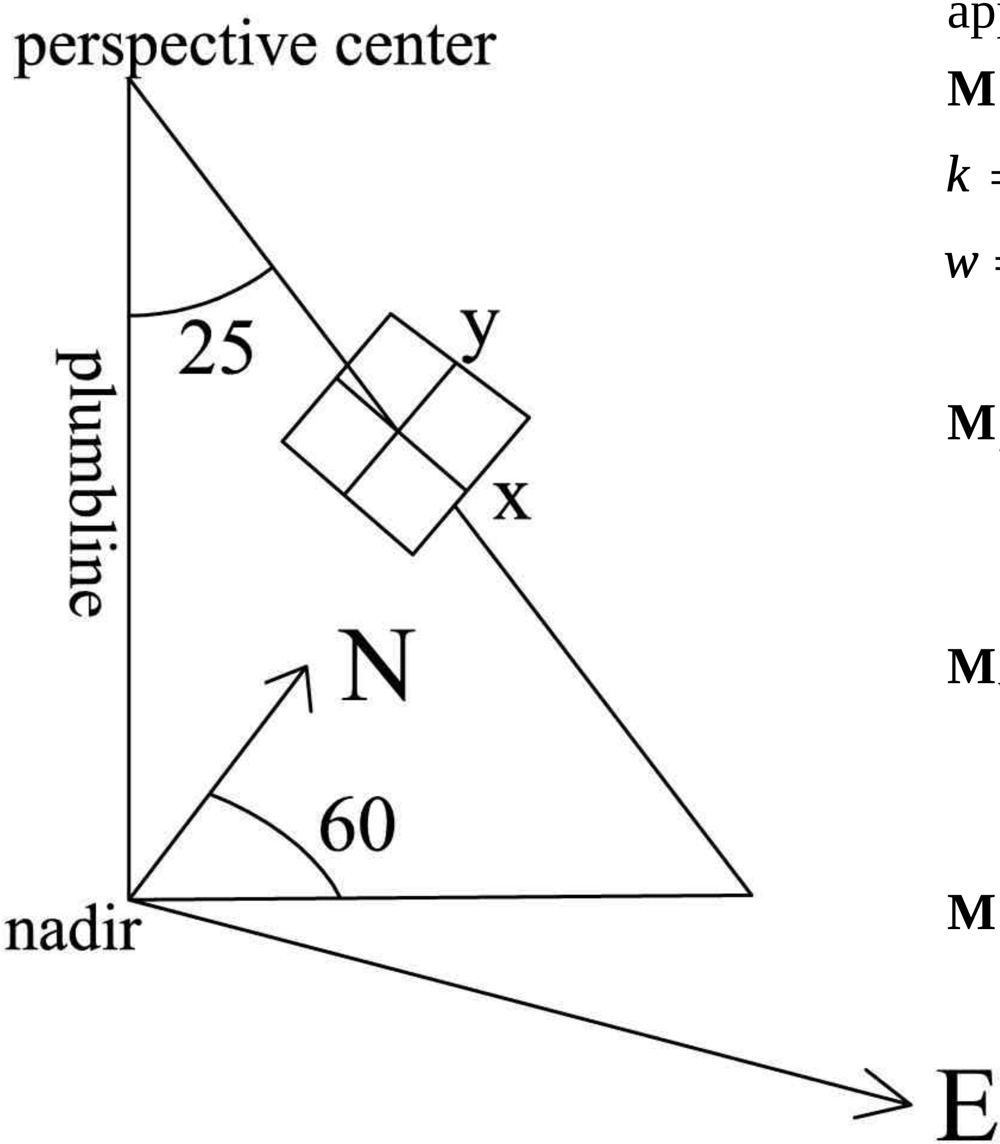
$$q_z = k \approx +90^\circ$$

$$\mathbf{M} = \mathbf{M}_k(90) =$$

$$\begin{bmatrix} \cos k & \sin k & 0 \\ -\sin k & \cos k & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 3, approximate the matrix



approximate using

$$\mathbf{M} = \mathbf{M_x}\mathbf{M_z} = \mathbf{M}_w \mathbf{M}_k$$

$$k = q_z \approx -60^\circ$$

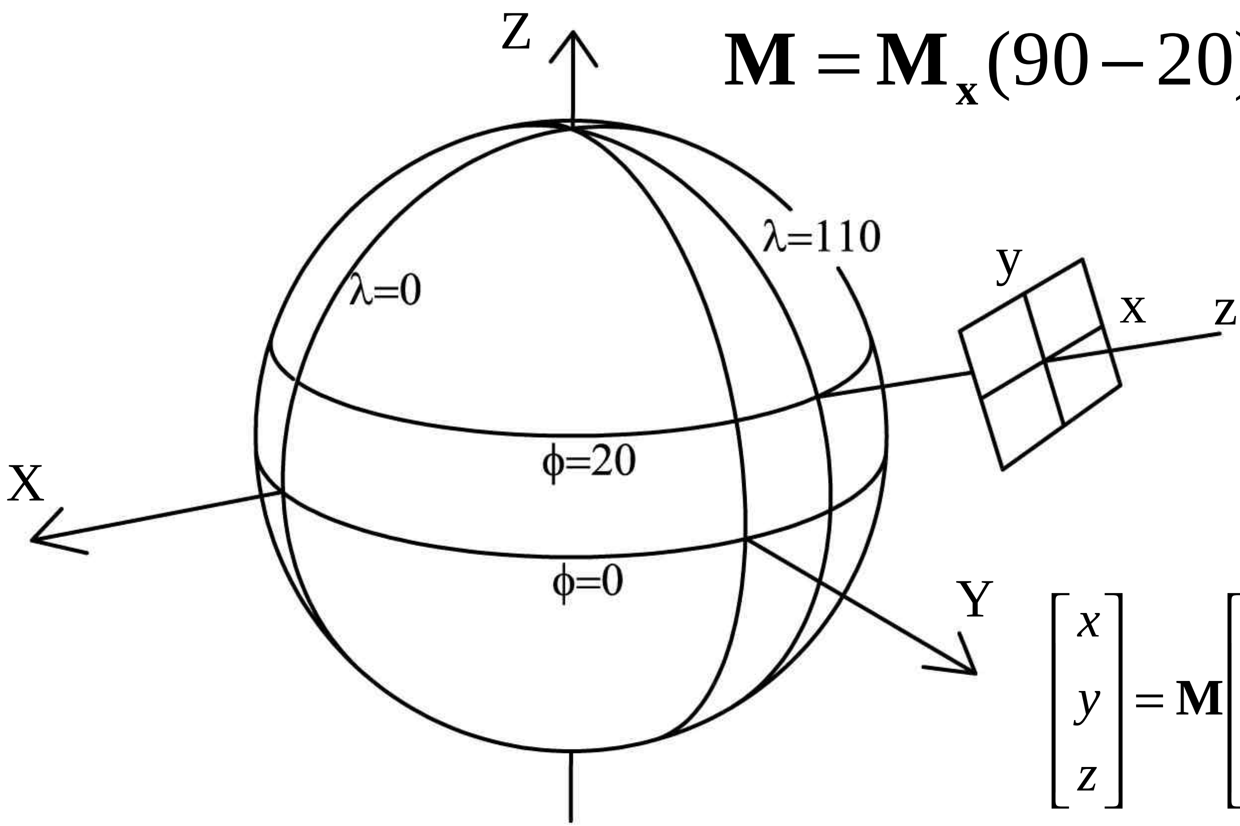
$$w = q_x \approx +25^\circ$$

$$\mathbf{M}_k = \begin{bmatrix} \cos(-60) & \sin(-60) & 0 \\ -\sin(-60) & \cos(-60) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{M}_w = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(25) & \sin(25) \\ 0 & -\sin(25) & \cos(25) \end{bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} 0.5000 & -0.8660 & 0 \\ 0.7849 & 0.4532 & 0.4226 \\ -0.3660 & -0.2113 & 0.9063 \end{bmatrix}$$

Example 4, approximate the matrix



$$\mathbf{M} = \mathbf{M}_{\mathbf{x}}(90-20)\mathbf{M}_{\mathbf{z}}(90)\mathbf{M}_{\mathbf{z}}(110)$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{M} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} -0.9397 & -0.3420 & 0 \\ 0.1170 & -0.3214 & 0.9397 \\ -0.3214 & 0.8830 & 0.3420 \end{bmatrix}$$

Extract the Angles for Order: wfk

$$M = \begin{bmatrix} \cos j \cos k & \cos w \sin k + \sin w \sin j \cos k & \sin w \sin k - \cos w \sin j \cos k \\ -\cos j \sin k & \cos w \cos k - \sin w \sin j \sin k & \sin w \cos k + \cos w \sin j \sin k \\ \sin j & -\sin w \cos j & \cos w \cos j \end{bmatrix}$$

$$f = \sin^{-1}(m_{31})$$

note : 2 possible values

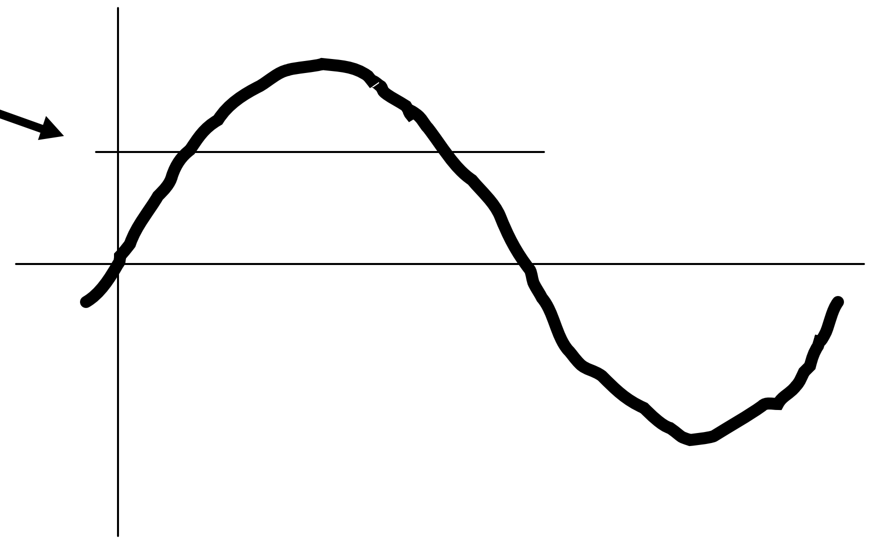
$$w = \tan^{-1}\left(\frac{-m_{32}/\cos f}{m_{33}/\cos f} \right)$$

note : use 2 - argument arctan

to get correct quadrant

$$k = \tan^{-1}\left(\frac{-m_{21}/\cos f}{m_{11}/\cos f} \right)$$

2 argument arctan



What if phi = 90 deg ?, i.e. cos 90 = 0

Special case: phi=90 deg

$$\mathbf{M} = \begin{bmatrix} 0 & \cos w \sin k + \sin w \cos k & \sin w \sin k - \cos w \cos k \\ 0 & \cos w \cos k - \sin w \sin k & \sin w \cos k + \cos w \sin k \\ 1 & 0 & 0 \end{bmatrix}$$

recall trig identities:

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$

$$\mathbf{M} = \begin{bmatrix} 0 & \sin(w+k) & -\cos(w+k) \\ 0 & \cos(w+k) & \sin(w+k) \\ 1 & 0 & 0 \end{bmatrix}$$

=> any pair of w, k with same sum yields same matrix

=> no unique solution for w, k

=> it is indeterminant or singular

=> also known as gimbal lock

Algebraic parameterization of rotation matrix

$$\mathbf{M} = \begin{bmatrix} d^2 + a^2 - b^2 - c^2 & 2(ab + cd) & 2(ac - bd) \\ 2(ab - cd) & d^2 - a^2 + b^2 - c^2 & 2(bc + ad) \\ 2(ac + bd) & 2(bc - ad) & d^2 - a^2 - b^2 + c^2 \end{bmatrix}$$

only 3 independent parameters so we need constraint that

$$a^2 + b^2 + c^2 + d^2 = 1$$

Claim: this one does not have singularities as exhibited by the euler angle or sequential rotation method, but also no easy interpretation of a,b,c,d.

Rotation about a directed line in space

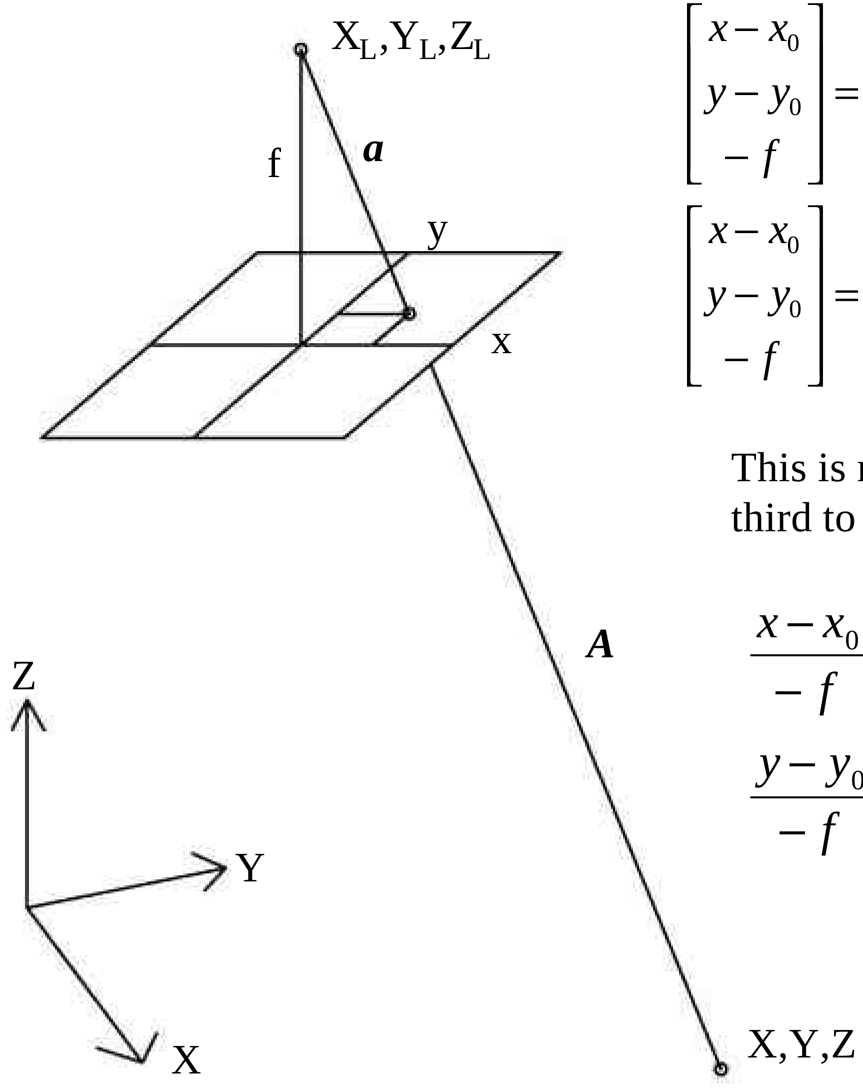
$$\mathbf{M} = \begin{bmatrix} a^{2}(1-\cos q)+\cos q & ab(1-\cos q)-g\sin q & ag(1-\cos q)+b\sin q \\ ab(1-\cos q)+g\sin q & b^{2}(1-\cos q)+\cos q & bg(1-\cos q)-a\sin q \\ ag(1-\cos q)-b\sin q & bg(1-\cos q)+a\sin q & g^{2}(1-\cos q)+\cos q \end{bmatrix}$$

must have only 3 independent parameters, can enforce unit length of line direction vector

$$a^{2}+b^{2}+g^{2}=1$$

Any rotation can be envisioned as a single rotation (theta) about a directed line in space. The direction of the line is given by its unit components: alpha, beta, gamma.

Collinearity Equations for Frame Sensor



$$\mathbf{a} = l\mathbf{A}$$

$$\begin{bmatrix} x - x_0 \\ y - y_0 \\ -f \end{bmatrix} = l\mathbf{M}\begin{bmatrix} X - X_L \\ Y - Y_L \\ Z - Z_L \end{bmatrix}$$

$$\begin{bmatrix} x - x_0 \\ y - y_0 \\ -f \end{bmatrix} = l\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}\begin{bmatrix} X - X_L \\ Y - Y_L \\ Z - Z_L \end{bmatrix}$$

This is really 3 equations, divide the first two by the third to eliminate the scale parameter, lambda

$$\frac{x - x_0}{-f} = \frac{m_{11}(X - X_L) + m_{12}(Y - Y_L) + m_{13}(Z - Z_L)}{m_{31}(X - X_L) + m_{32}(Y - Y_L) + m_{33}(Z - Z_L)}$$

$$\frac{y - y_0}{-f} = \frac{m_{21}(X - X_L) + m_{22}(Y - Y_L) + m_{23}(Z - Z_L)}{m_{31}(X - X_L) + m_{32}(Y - Y_L) + m_{33}(Z - Z_L)}$$

This works for ground points defined by coordinates, what if the “target” is defined only by direction?

Rearrange collinearity equations to give position in object space for given image point (with known height)

$$\begin{bmatrix} x - x_0 \\ y - y_0 \\ -f \end{bmatrix} = l\mathbf{M} \begin{bmatrix} X - X_L \\ Y - Y_L \\ Z - Z_L \end{bmatrix}$$

bring lambda and M left

$$\frac{1}{l}\mathbf{M}^{\mathrm{T}} \begin{bmatrix} x - x_0 \\ y - y_0 \\ -f \end{bmatrix} = \begin{bmatrix} X - X_L \\ Y - Y_L \\ Z - Z_L \end{bmatrix}$$

divide first two eqns by third

$$\frac{m_{11}(x - x_0) + m_{21}(y - y_0) + m_{31}(-f)}{m_{13}(x - x_0) + m_{23}(y - y_0) + m_{33}(-f)}(Z - Z_L) + X_L = X$$

$$\frac{m_{12}(x - x_0) + m_{22}(y - y_0) + m_{32}(-f)}{m_{13}(x - x_0) + m_{23}(y - y_0) + m_{33}(-f)}(Z - Z_L) + Y_L = Y$$

Directional Control

Divide the vector on the right by its length, R and multiply on the left to keep equality. Now the vector elements on the right are just direction cosines.

$$\begin{bmatrix} x - x_0 \\ y - y_0 \\ -f \end{bmatrix} = l R \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} (X - X_L)/R \\ (Y - Y_L)/R \\ (Z - Z_L)/R \end{bmatrix}$$

Rename them as (C_X,C_Y,C_Z)

$$\begin{bmatrix} x - x_0 \\ y - y_0 \\ -f \end{bmatrix} = l R \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} C_X \\ C_Y \\ C_Z \end{bmatrix}$$

Divide as before to eliminate the factor lambda-R

$$\frac{x - x_0}{-f} = \frac{m_{11} C_X + m_{12} C_Y + m_{13} C_Z}{m_{31} C_X + m_{32} C_Y + m_{33} C_Z}$$

$$\frac{y - y_0}{-f} = \frac{m_{21} C_X + m_{22} C_Y + m_{23} C_Z}{m_{31} C_X + m_{32} C_Y + m_{33} C_Z}$$

Now we have the image image points defined only in terms of external directions. This form can be used for stellar photogrammetry where it is more reasonable to use direction rather than coordinates (which would require *very* large numbers). The direction reference can be chosen in many ways. For stars, two reasonable systems would be (1) the tabulated Right Ascension and Declination, and (2) the local Azimuth and Elevation angles. (Some use Altitude angle instead of Elevation angle)

Use of collinearity equations to simulate a perspective view