Solving 2x2 systems

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

gauss elimination

$$3x + y = 7$$
$$x + 3y = 5$$

multiply 2nd equation by 3

$$3x + y = 7$$
$$3x + 9y = 15$$

subtract 1st equation from 2nd

$$8y = 8$$
$$y = 1$$

plug $y=1$ into equation 1

$$3x + 1 = 7$$
$$3x = 6$$
$$x = 2$$

Solution $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ ✓

cramer's rule:

$$x = \frac{\begin{vmatrix} 7 & 1 \\ 5 & 3 \end{vmatrix}}{\begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} 3 & 7 \\ 1 & 5 \end{vmatrix}}{\begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix}}$$

$$x = \frac{21-5}{9-1} = \frac{16}{8} = 2$$ ✓

$$y = \frac{15-7}{9-1} = \frac{8}{8} = 1$$

matrix inverse

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = A, \quad A^{-1} = \frac{\text{adjoint } A}{|A|}$$

adjoint A = [cofactor matrix]^T = C^T

$c_{ij} = (\pm 1)\cdot m_{ij}$, m_{ij} = minor = det submatrix minus row + col ij

$$C = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}, \quad C^T = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} = \text{adjoint } A$$

$$|A| = 9 - 1 = 8$$

$$A^{-1} = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}\cdot 1/8 = \begin{bmatrix} 3/8 & -1/8 \\ -1/8 & 3/8 \end{bmatrix}$$

check:

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 3/8 & -1/8 \\ -1/8 & 3/8 \end{bmatrix} = \begin{bmatrix} 9/8 - 1/8 & -3/8 + 3/8 \\ 3/8 - 3/8 & -1/8 + 9/8 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ ✓

now solve equation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3/8 & -1/8 \\ -1/8 & 3/8 \end{pmatrix}\begin{pmatrix} 7 \\ 5 \end{pmatrix} = \begin{pmatrix} 21/8 - 5/8 \\ -7/8 + 15/8 \end{pmatrix}$$

$$= \begin{pmatrix} 16/8 \\ 8/8 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$ ✓

* $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$